Elements of

SPHERICAL TRIGONOMETRY

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By

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PREFACE

This book has been planned and written for two purposes: for use as a class textbook, and for study and self-instruction without a teacher. There are very few independent self-contained books on spherical trigonometry now available, most treatments of the subject being contained in sketchy and supplementary chapters of books on plane trigonometry or in equally incomplete introductory chapters of scientific books which demand a knowledge of spherical trigonometry. In order to achieve this dual purpose, the book avoids both the sketchiness of these usual presentations and the detail of the complete theoretical treatise.

The book may be studied as a sequel to the author's "Textbook of Trigonometry" (plane) or any similar text, and also as an auxiliary or sequel to the shorter book on plane trigonometry in the author's series on "Mathematics for Self Study." It should be suitable for use in short courses given in summer sessions or evening sessions in colleges and universities, and also in courses given to students preparing for service or promotion in the naval and merchant marine services.

The introductory chapter sets forth the relation of spherical trigonometry to geometry and to plane trigonometry, indicates the background of previous training which the reader will require, and summarizes some of the fundamental definitions and properties of the trigonometric functions and of plane triangles which will be needed in the later chapters. If the book is used as a class text by properly prepared students the teacher may omit most of this chapter and use it only for reference as occasion requires. If the student requires a review of plane trigonometry or if he is studying without the aid of a teacher he may find a careful reading of this chapter to be useful as a review and introduction.

The second chapter presents a summary of certain results from elementary solid geometry and certain useful and important properties of the sphere and spherical triangles as a preparation for the detailed study of the third chapter. Chapter three then presents a unified and systematic development of all the necessary properties and formulas of spherical triangles in a manner which brings out the significant relations between them, reveals the continuity and connections of the entire subject, and reduces the time and space necessary for mastering the principles of the subject. This is in contrast to the piecemeal and disjointed development of the formulas as isolated results frequently given in parts at places where they are supposed to be first required. The author's teaching experience indicates that students get a better understanding of such relations and principles when they are presented as a unified complete sequence and as a natural development of the geometry of the sphere. The equations and formulas are then taken up in turn and adapted to numerical computation for the various cases of triangle solution in the next two chapters. Applications and the solution of a few illustrative technical problems are presented in the last chapter. In all numerical work care has been used to give simple and concise computation forms which have been found to be easily understood and reproduced by students. The use of cologarithms has been avoided for two reasons: students of elementary mathematics rarely have been taught their use and understanding, and since a subtraction must be made in their use in any case it is better made in written form than by an attempted mental operation and recording.

The purely formal algebraic derivations and the adaptations for numerical computation are given in full detail but the explanations and interpretations are made as direct and concise as possible. On the other hand the descriptions and explanations of the methods of reference and measurement on the terrestrial and celestial spheres are made full and informal. As most of the uses and applications of spherical trigonometry occur in connection with these

two spheres an understanding of their trigonometrical properties as expressed in terms of everyday observation is considered as being of the greatest importance for any one who has any occasion to study spherical trigonometry at all. The descriptions and explanations are made in such a way, however, as to require no special technical knowledge of any of the various subjects in which spherical trigonometry is to be used. Thus the book is made equally available and useful to prospective students of geodetic surveying, marine or aerial navigation, nautical or field astronomy, or general spherical and practical astronomy.

Besides the very efficient readers of the publisher and the printer only the author has read the proofs. The author and publisher will be glad to receive notices of any errors or misprints which may be discovered by readers.

J. E. Thompson.

Brooklyn, N. Y.
April, 1942.

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GREEK ALPHABET

Greek letter	Greek name	English equivalent	Greek letter	Greek name	English equivalent
Α α	Alpha	a	Ν ν	Nu	n
Β β	Beta	b	Ξ ξ	Xi	x
Γ γ	Gamma	g	Ο ο	Omicron	ŏ
Δ δ	Delta	d	Π π	Pi	p
Ε ε	Epsilon	ĕ	Ρ ρ	Rho	r
Ζ ζ	Zeta	z	Σ σ	Sigma	s
Η η	Eta	ē	Τ τ	Tau	t
Θ θ	Theta	th	Υ υ	Upsilon	u
Ι ι	Iota	i	Φ φ	Phi	ph
Κ κ	Kappa	k	Χ χ	Chi	ch
Λ λ	Lambda	l	Ψ ψ	Psi	ps
Μ μ	Mu	m	Ω ω	Omega	ō

ELEMENTS OF SPHERICAL TRIGONOMETRY

CHAPTER I

INTRODUCTION

1. *Historical Note.*—The word "trigonometry" is derived from the Greek words for *triangle* and *measure* (τριγωνον = *trigonon* or three-angle; and μετρειν = *metrein*, to measure). The science of *trigonometry* was originally a branch of geometry and had for its object the measurement of triangles and their angles. It was developed in its rudimentary form by the ancient Greeks and medieval Arabs in their applications of geometry to measurements in astronomy.

The celebrated Greek astronomer Hipparchus of Nicæa (born about 160 B.C.) is often referred to as the founder of trigonometry; and the Egyptian Greek astronomer Ptolemy of Alexandria (died about 168 A.D.) applied the principles of trigonometry also to stereographic projection, used in navigation, and wrote a great work on trigonometry and astronomy which has been handed down to us by the Arabs, who conquered Egypt in 641 A.D., as *The Almagest.* The Arabian astronomers and mathematicians extended the applications of trigonometry and introduced it into Europe when they extended the great Arabian Empire into Spain in the eighth century A.D.

Following the Revival of Learning (the *Renaissance*) in Europe during the thirteenth and fourteenth centuries, trigonometry was

hardly separated from the science of astronomy for several hundred years. It was finally established as a distinct branch of mathematics and developed into its modern form very largely through the work of one man, the great and famous Swiss mathematician Leonhard Euler (1707–1783), who also was largely responsible for the development of the *Calculus* after its discovery or invention by Newton and Leibnitz.

2. *Subject Matter of Trigonometry.*—Although originally devoted to the narrow subject of the measurement of triangles in astronomy, trigonometry has been developed along with the complete theory and general properties of angles until the measurement and properties of triangles constitute only a part of the complete subject. In modern pure and applied mathematics the theory of angles is of the highest importance in many places other than in measuring triangles.

Historically, trigonometry was divided into two parts: *plane trigonometry*, which dealt with triangles in a plane; and *spherical trigonometry*, which dealt with triangles formed on the surfaces of spheres. In modern times, a third division of the subject has been developed: *analytical trigonometry*, which deals with the algebraic theory of the entire subject.

The modern science of *plane trigonometry* includes the general definitions and properties of plane angles, angle functions and triangles, and their uses in elementary plane and solid geometry, mechanics, physics, and engineering. The elements of plane trigonometry are studied in high school or elementary college courses and treatments suitable for most ordinary purposes are to be found in any good high school text on this subject or in the elementary book named at the end of this chapter.

The subject of *analytical trigonometry*, together with some more advanced parts of plane trigonometry, is treated in any complete college textbook on the subject, such as the one mentioned at the end of this chapter, and its results are used and applied in all higher mathematics.

The present book deals with the elementary parts of *spherical trigonometry* and a few illustrative applications in a manner suitable for readers who may study later such subjects as mathematical geography, geodetical surveying, marine or aerial navigation, nautical astronomy, or spherical astronomy.

3. *Prerequisites for the Study of Spherical Trigonometry.*—The subject matter of *spherical trigonometry* consists of the properties, measurement, and calculation of angles and triangles formed by intersecting curved lines drawn in a particular manner on the surfaces of spheres, and the advanced parts of the subject deal with the uses and applications of such angles and triangles in the technical subjects mentioned above.

In order to study spherical trigonometry, therefore, the student should have a knowledge of plane geometry, some parts of solid geometry, plane trigonometry, and elementary algebra, as these are usually taught in the high schools. The following chapter of this book contains a summary of some of the necessary parts of solid geometry and the geometry of the sphere. This summary may be enlarged by the teacher or by the student's reading in such books as those mentioned at the end of this chapter.

For review and reference purposes the remainder of this chapter presents concise statements of some of the chief definitions and formulas of elementary plane trigonometry.

4. *Plane Angles and Angle Measure.*—A *plane angle* is the figure formed by two straight lines of any lengths which meet at a point or are drawn from one point. The two lines are called the *sides* (or *arms*) of the angle, and the point of intersection the *vertex* of the angle. The *size* or magnitude of such an angle is the amount of opening, separation, or spread between its two sides, the sharpness or bluntness of the "corner" formed at the vertex, and has nothing to do with the length of the sides.

In Fig. 1 the two lines AB, AC, EF, EG, meeting at A, E form the angles BAC, FEG, with sides AB, AC, EF, EG, and vertices A, E. These lines may be of any length. These angles are desig-

nated as $\angle BAC$, $\angle FEG$. The symbol $\measuredangle$ represents "angles." $\angle FEG$ in (*b*) is larger than $\angle BAC$ in (*a*).

If two angles have one side in common, and that side between their other two sides, as in Fig. 2, the two angles are said to be *ad-*

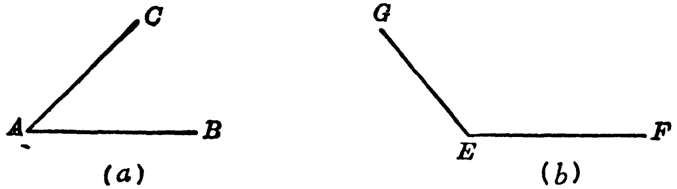


Fig. 1.

jacent angles. Thus, in Fig. 2, $\measuredangle BAC$, CAD are adjacent angles, and so are $\measuredangle FEG$, GEH. The angle BAD in (*a*) with sides AB, AD is the *sum* of the two $\measuredangle BAC$, CAD. This is written $\angle BAD = \angle BAC + \angle CAD$. Similarly, in (*b*), $\angle FEH = \angle FEG + \angle GEH$.

When the two external (not common) sides of two adjacent angles lie in one straight line the two angles are said to be *supple-*

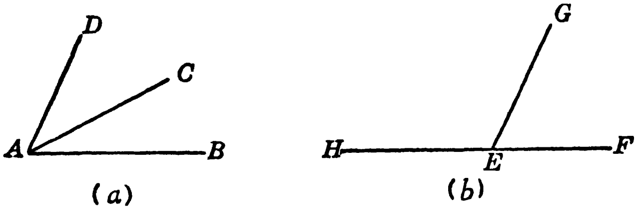


Fig. 2.

mentary. Thus, in Fig. 2(*b*), EF and EH form the single straight line HF and $\measuredangle FEG$, GEH are supplementary. Each of two supplementary angles is the *supplement* of the other. Thus, $\angle GEH$ is the supplement of $\angle FEG$, and $\angle FEG$ is the supplement of $\angle GEH$. The sum, $\angle FEH$, is called a *straight angle*.

When two lines meet at a point and extend both ways from that point as in Fig. 3, they form four angles. The two "opposite"

angles are called *vertical* angles and it is proved in geometry that vertical angles are equal to each other. Thus, in Fig. 3(*a*), $\measuredangle BAC$, DAE are vertical angles and $\angle BAC = \angle DAE$. Also $\measuredangle CAD$, BAE are vertical and $\angle CAD = \angle BAE$. The same is true of the opposite pairs (verticals) in (*b*). Thus $\angle XOY = \angle X'OY'$ and $\angle YOX' = \angle XOY'$.

When *all four* of the angles formed by two intersecting lines are *equal to one another* the two lines are said to be *perpendicular* to

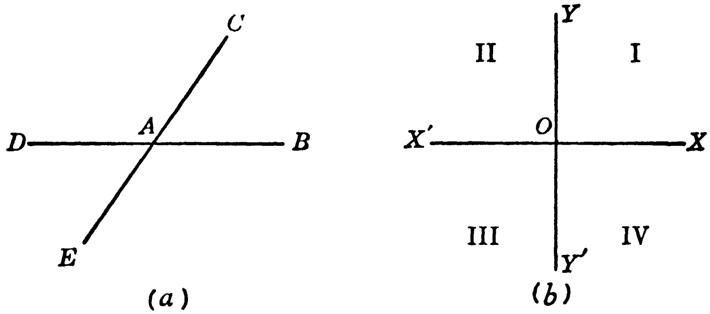


Fig. 3.

each other and each of the four equal angles is called a *right angle.* Thus in Fig. 3(*b*) the line $X'X$ is perpendicular to line $Y'Y$, and the four angles are all equal: $\angle XOY = \angle YOX' = \angle X'OY' = \angle Y'OX$. Perpendicularity of two lines is indicated by the sign $\perp$. Thus $XX' \perp YY'$. (A single line by itself cannot be perpendicular; two are required and each is perpendicular *to* the other.)

An angle which is smaller than a right angle is called an *acute* angle, and one which is greater than a right angle but less than the sum of two right angles (or a straight angle) is an *obtuse* angle. Thus in Fig. 3(*a*) $\measuredangle BAC$, DAE are acute angles, and $\measuredangle CAD$, BAE are obtuse.

If the sum of two acute angles is equal to a right angle the two angles are said to be *complementary* and each is the *complement* or *co-angle* of the other.

The definitions given above are taken from elementary geometry. In trigonometry these definitions are extended to apply to angles of any magnitude. The angles of Fig. 1 are drawn below as Fig. 4.

Here the lines AB and AC are not considered as two lines drawn separately from A to B (or B to A) and from A to C (or C to A), but AC is considered as being *rotated* from the position AB to the position AC about A as a pivot point. The angle BAC

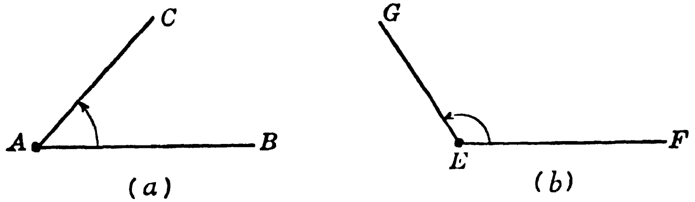


FIG. 4.

is then said to be *generated* by rotation of the line about the vertex A from its *initial* position AB to its *terminal* position AC. Similarly in (*b*), $\angle FEG$ is generated by the rotation of its generating line from EF to EG about the vertex E.

The measure of the size or magnitude of an angle is then the *amount of rotation* about its vertex which is required to generate the angle. Thus in Fig. 4 the line EG is rotated farther than the line AC and $\angle FEG$ is greater than $\angle BAC$.

The two perpendicular lines $X'X$ and $Y'Y$ of Fig. 3(*b*) are called *coordinate axes* and are the lines used for reference in plotting points and drawing graphs in algebra and analytical geometry. $X'X$ is called the axis of *abscissas* and $Y'Y$ the axis of *ordinates*. The point of intersection O is called the coordinate *origin*. Distances or lengths measured on or parallel to $X'X$ from O or $Y'Y$ to the right are given the sign plus (+) and called *positive*; to the left minus (−) or *negative*. Similarly, distances or lengths measured on or parallel to $Y'Y$ from O or $X'X$ upward are positive (+);

and downward negative (−). The axes divide their plane into four *quadrants*. These are numbered I, II, III, IV as in Fig. 3(*b*) and are called respectively the *first*, *second*, *third*, and *fourth quadrants*.

When an angle is drawn, placed, or imagined as placed on the coordinate axes with its vertex *at the origin*, its initial side on the *positive abscissa* axis, and its terminal side anywhere in the plane of the axes, the angle is said to be in the *standard position*. Thus

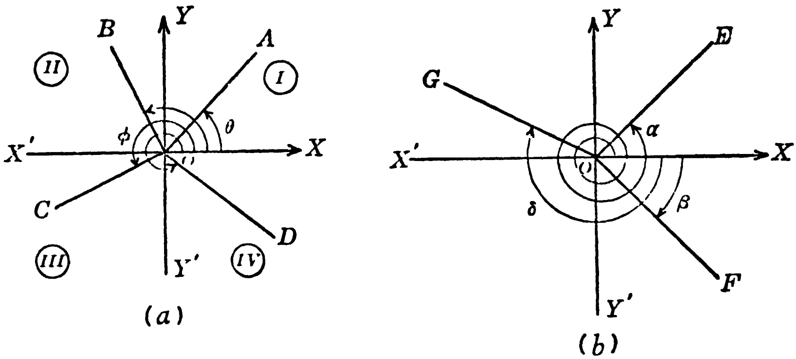


FIG. 5.

in Fig. 5(*a*) the ∡*XOA*, *XOB* are in the standard position. These may be thought of as the ∡*BAC*, *FEG* of Fig. 4 placed in the standard position.

If the terminal side of an angle in standard position lies in the first quadrant, as ∠*XOA* in Fig. 5(*a*), the angle is said to be "in the first quadrant" or to be a "first quadrant angle." Thus, any acute angle is a first quadrant angle. Similarly all obtuse angles are in the second quadrant, as ∠*XOB*. A third quadrant angle, as ∠*XOC*, or one in the fourth quadrant, as ∠*XOD*, is called a *reflex* angle; and an angle generated by a complete rotation, or four right angles, is called a *complete angle* or a *perigon*.

In Fig. 5 the positive ends of the coordinate axes are marked with an arrow-head to indicate the positive directions and to dis-

tinguish these lines from all other lines. Similarly, curved arrows are used to indicate the rotation or magnitude of angles, and symbols are used (as ordinary algebraic symbols) to represent the angle and the measure of its magnitude. Greek letters are generally used to represent angles as italic letters (a, b, c, x, y, etc.) are used to represent other numbers, lengths, etc. (The Greek alphabet is printed in the front of this book.) Thus in Fig. 5(a) the $\angle XOA = \theta = \frac{1}{2}$ rt. $\angle$ and $\angle XOC = \phi = 2\frac{1}{3}$ rt. $\measuredangle$.

Angles may be of any magnitude whatever; thus in Fig. 5(b), the angle α with terminal side OE in the first quadrant is generated by rotation through $2\frac{1}{8}$ complete revolutions. Its magnitude is therefore $\alpha = 2\frac{1}{8}$ complete $\measuredangle = 8\frac{1}{2}$ rt. $\measuredangle$.

Angles may also be positive or negative. A *positive angle* is generated by rotation from the standard initial position (OX) *upward* toward the positive OY and about O in the counter-clockwise direction. A negative angle is generated by rotation from OX *downward* toward the negative OY' in the clockwise direction about O. Thus the angles in Fig. 5(a) and the angle α in (b) are positive, while the angles β and δ in (b) are negative. The algebraic values of β and δ are $\beta = -\frac{1}{2}$ rt. $\angle$ and $\delta = -2\frac{1}{3}$ rt. $\measuredangle$. When no sign is given the angle is to be considered positive, as in the ordinary use of signs in algebra. Algebraic operations with positive and negative angles follow all the rules of signs of algebra.

5. *Units of Angle Measure.*—As angles are measured by rotation, a complete revolution or circle is taken as the basic unit of angle measure. Thus a circle of any radius is drawn with its center at the coordinate origin and the generating line (or side) of the angle is then a rotating radius of the circle. When the vertex of an angle is at the center of a circle it is called a *central angle* and it is proved in geometry that "a central angle is measured by the arc of the circle intercepted between its sides." This arc is the arc swept over by a point on the generating radius and so the length of the arc is a measure of the rotation which generates the angle. Using this arc as the measure of the central angle in any

circle, the following are definitions of the standard units of angle measure:

(*a*) The central angle subtended (measured) by an arc equal to *one-fourth of the circumference* is called a *quadrant.*

(*b*) The central angle subtended by an arc equal to *one-sixth of the circumference* is called a *sextant.*

(*c*) The central angle subtended by an arc *equal in length to the radius* is called *a radian.*

These definitions may be stated in another manner for comparison, as follows:

(*a*) The *quadrant* is the central *right angle* subtended by the arc intercepted by a side of the inscribed square as a *chord.*

(*b*) The *sextant* is the central acute angle subtended by the arc intercepted by a side of the inscribed regular hexagon as a *chord equal to the radius.*

(*c*) The *radian* is the central acute angle subtended by the *arc equal to the radius.*

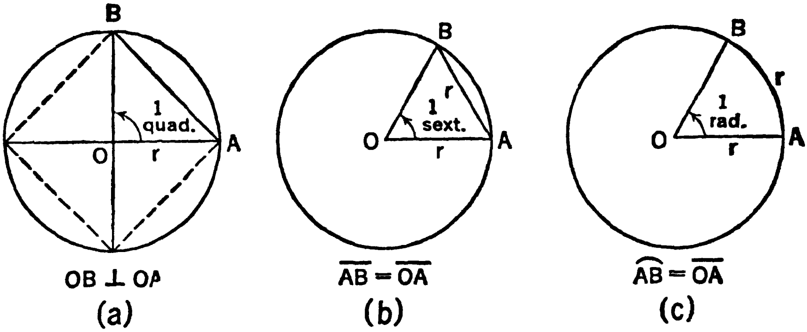


FIG. 6.

These definitions are illustrated in Fig. 6. In

(*a*) $OB \perp OA$ and $\angle AOB$ is one *quadrant;*

(*b*) chord $\overline{AB} = \overline{OA} = r$ and $\angle AOB$ is one sextant;

(*c*) arc $\widehat{AB} = \overline{OA} = r$ and $\angle AOB$ is one *radian.*

A circle or complete angle (one revolution) obviously contains and is equal to four quadrants, and to six sextants. Also the circumference is equal to 2π times the radius ($C=2\pi r$), and the circle therefore contains $2\pi=6.2832+$ radians of angle. Therefore:

$$1 \text{ circle}=4 \text{ quadrants}=6 \text{ sextants}=2\pi \text{ radians},$$

or

$$2 \text{ quadrants}=3 \text{ sextants}=\pi \text{ radians}.$$

The sextant is divided into sixty equal parts called *degrees* (°), the degree again into 60 equal parts called *minutes* (′), and the minute into 60 *seconds* (″). Thus

$$1 \text{ circle}=6 \text{ sext.}=360°=21600'=1296000''.$$

As π radians $=3$ sextants $=180$ degrees,

$$1 \text{ radian}=\frac{180}{\pi}=57.2957795 \text{ deg.}$$

$$1 \text{ degree}=\frac{\pi}{180}=0.0174533 \text{ rad.}$$

Radians are divided into tenths, hundredths, thousandths, etc., and expressed decimally. As π is a non-terminating decimal the decimal equivalents given above are not exact. Approximately, therefore, when

$$\pi \text{ rad.}=180 \text{ deg.},$$

$$1 \text{ rad.}=57°17'45'',$$

$$1 \text{ deg.}=.0174533 \text{ rad.}$$

The system of angle measure using degrees, minutes, and seconds, is called *sexagesimal* measure. The system using radians and decimal subdivisions is called *natural* or *circular* measure.

When θ is the circular measure of any central angle in a circle

of radius r and circumference $C=2\pi r$, and s is the length of the arc subtended by the angle, then the entire angle about the center is 2π radians, and $\theta : 2\pi :: s : C$, or $\frac{\theta}{2\pi}=\frac{s}{2\pi r}$. Therefore:

$$\theta=\frac{s}{r}, \quad s=r\theta. \tag{1}$$

This important relation is of frequent use in work pertaining to circles and spheres.

The same systems and units of angle measure are used in spherical trigonometry as in plane trigonometry.

6. *The Trigonometric or Circular Functions.*—Consider an angle θ, in circular or sexagesimal measure, in standard position on the XY coordinate axes, as in Fig. 7 below, the terminal side lying in any quadrant, as in (a), (b), (c), or (d). Choose a point P on the terminal side of the angle at any distance $\overline{OP}=r$ from the center measured positively outward from O to P, and draw $PM\perp OX$. Then $\overline{OM}=x$ is the *abscissa*, $\overline{MP}=y$ the *ordinate*, and $\overline{OP}=r$ the *radius vector* of the point P. x and y together are called the *coordinates* of P and the point is denoted by $P(x, y)$. x is positive for points in the first and fourth quadrants and negative in the second and third. y is positive for points in the first and second quadrants and negative in the third and fourth. Thus $P(2, 3)$ is in the first quadrant, $(-4, 7)$ is in the second, $(-5, -3)$ in the third, and $(6, -2)$ in the fourth quadrant.

As the value of the angle θ varies, the coordinates x, y vary, and for any specified value of θ the values of x, y vary as P is moved along the terminal side of the angle, but for any *particular angle*, with the terminal side fixed in position, the *ratios* $\frac{x}{r}, \frac{y}{r}, \frac{y}{x}$ and their reciprocals $\frac{r}{x}, \frac{r}{y}, \frac{x}{y}$ *do not vary*. These ratios do vary, however, when the angle is varied. The values of these ratios

therefore *depend on the value of the angle* and hence are *functions* of the angle.

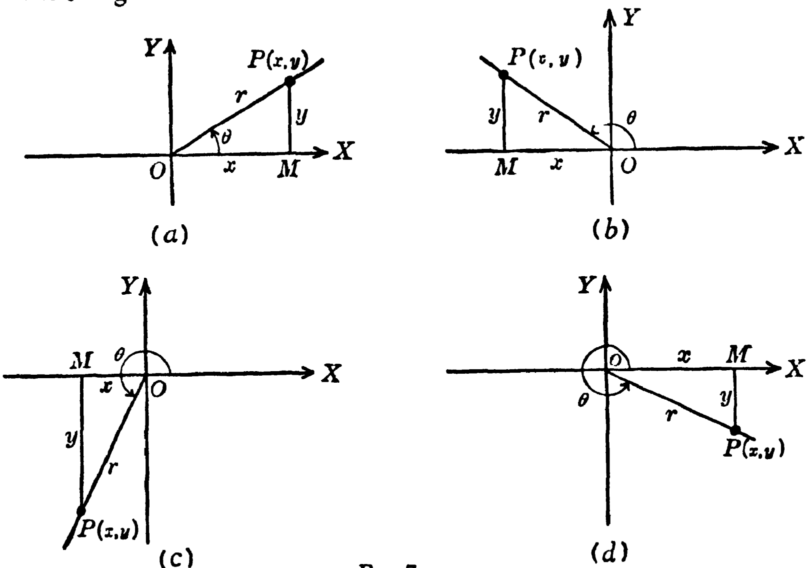


FIG. 7.

These six function ratios are of the greatest importance and each is given a special name. In any quadrant, for any angle θ, and for any point P on the terminal side:

(*a*) The ratio $\frac{y}{r}$ or (ordinate) $\div$ (radius) is called the *sine* of the angle θ.

(*b*) The ratio $\frac{y}{x}$ or (ordinate) $\div$ (abscissa) is called the *tangent* of the angle θ.

(*c*) The ratio $\frac{r}{x}$ or (radius) $\div$ (abscissa) is called the *secant* of the angle θ.

These three names are abbreviated as *sin*, *tan*, *sec*, respectively,

and the three definitions are written in the form of equations or formulas as follows:

$$\sin\theta=\frac{y}{r},\quad \tan\theta=\frac{y}{x},\quad \sec\theta=\frac{r}{x}. \tag{2a}$$

The reciprocals of these three ratios are defined and named as follows:

(*d*) The ratio $\frac{x}{r}$ or (abscissa) ÷ (radius) is called the *cosine* of the angle θ.

(*e*) The ratio $\frac{x}{y}$ or (abscissa) ÷ (ordinate) is called the *cotangent* of the angle θ.

(*f*) The ratio $\frac{r}{y}$ or (radius) ÷ (ordinate) is called the *cosecant* of the angle θ.

These last three names are abbreviated as *cos*, *cot* (or *ctn*), *csc* (or *cosec*), respectively, and these three definitions are written as the formulas

$$\cos\theta=\frac{x}{r},\quad \cot\theta=\frac{x}{y},\quad \csc\theta=\frac{r}{y}. \tag{2b}$$

Because of the changes in sign of x and y from quadrant to quadrant (r is always taken as positive, outward), the algebraic rules of signs applied to the function definition formulas (2) give different signs in the several quadrants. These are shown in the following table.

SIGNS OF THE FUNCTIONS

Funct. / Quad	sin	cos	tan	cot	sec	csc
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

In each case a co-function of any angle is the corresponding function of the complementary angle or co-angle. These relations are shown in the following formulas.

$$\left.\begin{aligned} \cos\theta &= \sin(90^\circ-\theta), & \sin\theta &= \cos(90^\circ-\theta), \\ \cot\theta &= \tan(90^\circ-\theta), & \tan\theta &= \cot(90^\circ-\theta), \\ \csc\theta &= \sec(90^\circ-\theta); & \sec\theta &= \csc(90^\circ-\theta). \end{aligned}\right\} \tag{3}$$

In addition to the six trigonometric functions defined above, four others are defined as follows: The

(*g*) versed sine $= 1 -$ cosine

(*h*) coversed sine $= 1 -$ sine

(*i*) suversed sine $= 1 +$ cosine $= 2 -$ sine

(*j*) haversine $=$ (versed sine) $\div 2 = \frac{1}{2}(1 -$ cosine)

These are abbreviated, for any angle θ, as

$$\left.\begin{aligned} \operatorname{ver}\theta &= 1-\cos\theta \\ \operatorname{cov}\theta &= 1-\sin\theta \\ \operatorname{suv}\theta &= 1+\cos\theta \\ \operatorname{hav}\theta &= \frac{1-\cos\theta}{2} \end{aligned}\right\} \tag{4}$$

These four functions are rarely used in many fields of applied mathematics and are ordinarily not studied in trigonometry but in the special subject in which they are used. The *haversine* (half-versine) is studied and used in navigation and astronomy.

Numerical values of the functions are found by geometrical construction and calculation or by means of special computing formulas which are developed in analytical trigonometry. These values are called the *natural functions* and are listed in special

tables in decimal form to four, five, six, seven, or more decimal places.

The common logarithms (base 10) of the function numbers are needed in many calculations, and are called the *logarithmic functions*. These are listed in tables similar to those of the natural functions, and to the same numbers of decimal places. For many uses in engineering, navigation, and nautical astronomy 5-place tables are suitable and will be used in this book. For more precise work in these subjects and in geodetical surveying and spherical astronomy, tables of six, seven, or more places are used. Such tables are generally accompanied by instructions for their use and it is assumed here that the reader knows how to use the tables.

In describing angle measure in Arts. 4 and 5 and in defining the trigonometric functions above, it is stated that any radius r may be used. If $r=1$ (one centimeter, inch, foot, yard, meter, etc., i.e., *unity* in any system of linear measure) the definitions and values of the functions are easily represented by certain lines drawn in connection with the circle, and hence the name *circular functions*. A circle of radius $r=1$ (unity) is called the *unit circle.*

In Fig. 8 below each of the four circles is a unit circle, i.e., in each case $\overline{OA}=\overline{OB}=\overline{OP}=1$, and $\angle AOP=\theta$ is an angle in standard position. The terminal side meets the circle at P and is extended (forward through P or backward through O) to meet the tangent lines drawn at A and B, the ends of the *positive* axes (radii) OA and OB. Then in each quadrant

$$\sin\theta=\frac{MP}{OP}=\frac{MP}{1}=\overline{MP}$$

$$\cos\theta=\frac{OM}{OP}=\frac{OM}{1}=\overline{OM}$$

$$\tan\theta=\frac{MP}{OM}=\frac{AT}{OA}=\frac{AT}{1}=\overline{AT}$$

$$\cot \theta = \frac{OM}{MP} = \frac{BT'}{OB} = \frac{BT'}{1} = \overline{BT'}$$

$$\sec \theta = \frac{OP}{OM} = \frac{OT}{OA} = \frac{OT}{1} = \overline{OT}$$

$$\csc \theta = \frac{OP}{MP} = \frac{OT'}{OB} = \frac{OT'}{1} = \overline{OT'}$$

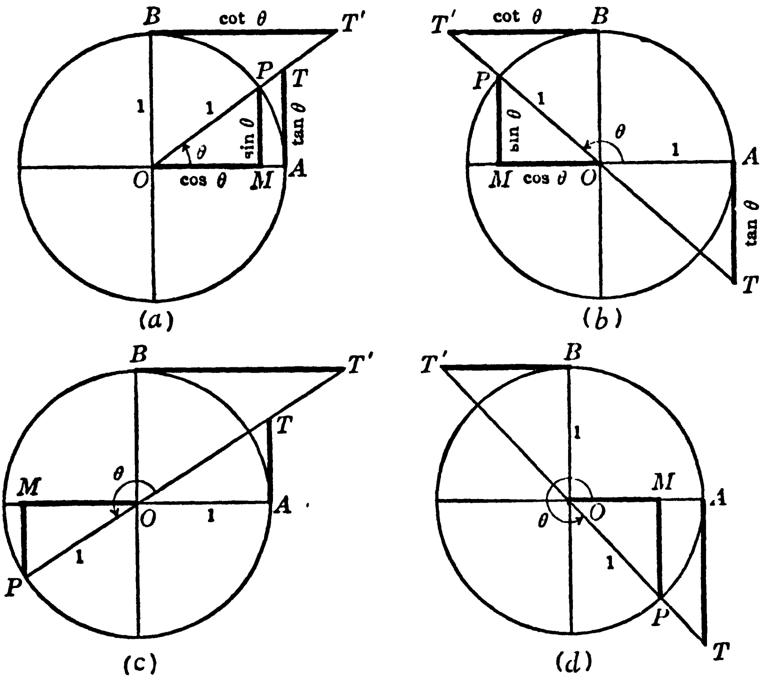


FIG. 8.

In every case for each angle $AOP = \theta$, the four chief functions $\sin \theta = \overline{MP}$, $\cos \theta = \overline{OM}$, $\tan \theta = \overline{AT}$, $\cot \theta = \overline{BT'}$ are shown as heavy black lines. The lengths of these lines give the magnitudes of these functions (to scale $\overline{OP} = 1$) and the directions give their

algebraic signs. The functions $\sec\theta=\overline{OT}$ and $\csc\theta=\overline{OT'}$ are also given in magnitude (length to scale $\overline{OP}=1$) and sign, forward through P (+) or backward through O (−).

In accordance with the definitions (4), in each circle in Fig. 8, $\operatorname{ver}\theta=\overline{MA}$ and $\operatorname{hav}\theta=\frac{1}{2}\,\overline{MA}$. The coversed sine and suversed sine are not shown in Fig. 8.

7. *Properties of the Functions.*—From the function definition formulas (2) it is seen at once that

$$\csc\theta=\frac{r}{y}=\frac{1}{\left(\dfrac{y}{r}\right)}=\frac{1}{\sin\theta},\quad\text{and}\quad\sin\theta=\frac{1}{\csc\theta};$$

$$\sec\theta=\frac{r}{x}=\frac{1}{\left(\dfrac{x}{r}\right)}=\frac{1}{\cos\theta},\quad\text{and}\quad\cos\theta=\frac{1}{\sec\theta};$$

$$\cot\theta=\frac{x}{y}=\frac{1}{\left(\dfrac{y}{x}\right)}=\frac{1}{\tan\theta},\quad\text{and}\quad\tan\theta=\frac{1}{\cot\theta}.$$

These six formulas may be combined as

$$\sin\theta\cdot\csc\theta=1,\quad\cos\theta\cdot\sec\theta=1,\quad\tan\theta\cdot\cot\theta=1. \tag{5}$$

Another important relation is also immediately obvious from (2):

$$\tan\theta=\frac{\sin\theta}{\cos\theta},\quad\cot\theta=\frac{\cos\theta}{\sin\theta}. \tag{6}$$

In the unit circle we also have in the right triangles OMP, OAT, OBT';

$$\overline{OM}^2+\overline{MP}^2=\overline{OP}^2,\quad\overline{OT}^2-\overline{AT}^2=\overline{OA}^2,\quad\overline{OT'}^2-\overline{BT'}^2=\overline{OB}^2,$$

or

$$\left.\begin{aligned}\cos^2\theta+\sin^2\theta&=1,\\ \sec^2\theta-\tan^2\theta&=1,\\ \csc^2\theta-\cot^2\theta&=1.\end{aligned}\right\} \tag{7}$$

The following important formulas are given here without proof, for later use and reference. They are derived in textbooks on plane trigonometry. In these formulas θ and ϕ are any two angles, measured in circular or sexagesimal units.

$$\left.\begin{aligned}\sin(\theta\pm\phi)&=\sin\theta\cos\phi\pm\cos\theta\sin\phi,\\ \cos(\theta\pm\phi)&=\cos\theta\cos\phi\mp\sin\theta\sin\phi.\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}\tan(\theta\pm\phi)&=\frac{\tan\theta\pm\tan\phi}{1\mp\tan\theta\tan\phi},\\ \cot(\theta\pm\phi)&=\frac{\cot\theta\cot\phi\mp1}{\cot\theta\pm\cot\phi}.\end{aligned}\right\}\tag{9}$$

$$\left.\begin{aligned}\sin 2\theta&=2\sin\theta\cos\theta,\\ \cos 2\theta&=\cos^2\theta-\sin^2\theta\\ &=2\cos^2\theta-1\\ &=1-2\sin^2\theta.\end{aligned}\right\}\tag{10}$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta},\qquad \cot 2\theta=\frac{\cot^2\theta-1}{2\cot\theta}.\tag{11}$$

$$\left.\begin{aligned}\sin 3\theta&=3\sin\theta-4\sin^3\theta,\\ \cos 3\theta&=4\cos^3\theta-3\cos\theta.\end{aligned}\right\}\tag{12}$$

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}},\quad \cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}},\tag{13}$$

the sign ($\pm$) being determined by the quadrant in which $\frac{\theta}{2}$ lies.

$$\left.\begin{aligned}\tan\frac{\theta}{2}&=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}=\csc\theta-\cot\theta,\\ \cot\frac{\theta}{2}&=\frac{\sin\theta}{1-\cos\theta}=\frac{1+\cos\theta}{\sin\theta}=\csc\theta+\cot\theta.\end{aligned}\right\}\tag{14}$$

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},\quad \cot\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}},\tag{15}$$

the sign ($\pm$) being determined by the quadrant in which $\frac{\theta}{2}$ lies.

$$\left.\begin{aligned}\sin\theta+\sin\phi&=2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),\\ \sin\theta-\sin\phi&=2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right).\end{aligned}\right\}\tag{16}$$

$$\left.\begin{aligned}\cos\phi+\cos\theta&=2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),\\ \cos\phi-\cos\theta&=2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right).\end{aligned}\right\}\tag{17}$$

Functions of Related Angles

	$-\theta$	$90^\circ\pm\theta$	$180^\circ\pm\theta$	$270^\circ\pm\theta$	$n(360^\circ)\pm\theta$
sin	$-\sin\theta$	$+\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos	$+\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$+\cos\theta$
tan	$-\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
cot	$-\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$
sec	$+\sec\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$	$+\sec\theta$
csc	$-\csc\theta$	$+\sec\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$

The following series formulas hold good only when the angle (θ) is measured in radians:

$$\left.\begin{aligned}\sin\theta &= \theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}-\frac{\theta^7}{5040}+\cdots,\\ \cos\theta &= 1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\frac{\theta^6}{720}+\cdots,\\ \tan\theta &= \theta+\frac{\theta^3}{3}+\frac{2\theta^5}{15}+\frac{17\theta^7}{315}+\cdots.\end{aligned}\right\} \tag{18}$$

8. *Formulas of Plane Right Triangles.*—In the first quadrant, x, y, r are all positive, and an acute angle is an angle of the first

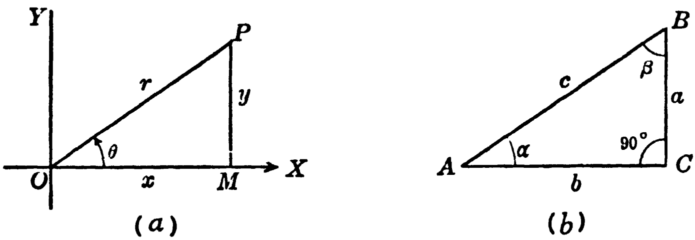


FIG. 9.

quadrant. By the definition formulas (2) and (3), therefore, all the functions of the interior angles of a plane right triangle are positive. All calculations pertaining to a single plane right triangle may therefore be carried out without considering algebraic signs or coordinate reference axes. The triangle OMP of Fig. 7(a), which is reproduced above as Fig. 9(a), may therefore be considered as removed from the coordinate axes and drawn separately as in Fig. 9(b). The lettering and marking used in this figure (b) is standard and should always be used for isolated plane right triangles.

In this figure, A, B, C represent points (vertices) and not angles. The angles of the triangle are α, β, 90°, and both α, β are acute. As their sum is a right angle (90°) they are complementary (co-angles).

The function definition formulas as applied to right $\triangle ABC$ therefore become:

$$\left.\begin{aligned} \sin\alpha &= \frac{y}{r} = \frac{\text{leg opposite}}{\text{hypotenuse}} = \frac{a}{c}, \\ \cos\alpha &= \frac{x}{r} = \frac{\text{leg adjacent}}{\text{hypotenuse}} = \frac{b}{c}, \\ \tan\alpha &= \frac{y}{x} = \frac{\text{leg opposite}}{\text{leg adjacent}} = \frac{a}{b}, \\ \cot\alpha &= \frac{x}{y} = \frac{\text{leg adjacent}}{\text{leg opposite}} = \frac{b}{a}, \end{aligned}\right\} \tag{19}$$

with corresponding formulas for sec α and csc α; and for angle β,

$$\left.\begin{aligned} \sin\beta &= \sin(90^\circ-\alpha) = \cos\alpha = \frac{b}{c}, \\ \cos\beta &= \cos(90^\circ-\alpha) = \sin\alpha = \frac{a}{c}, \\ \tan\beta &= \tan(90^\circ-\alpha) = \cot\alpha = \frac{b}{a}, \\ \cot\beta &= \cot(90^\circ-\alpha) = \tan\alpha = \frac{a}{b}. \end{aligned}\right\} \tag{20}$$

When any two sides or any side and acute angle of the right triangle are known, the remaining three parts can always be calculated by using the appropriate two of the formulas (19), (20) with the relation $\alpha+\beta=90^\circ$. Thus, solving the formulas (19), (20) for each of the quantities they contain in terms of the other two we have:

$$\left.\begin{array}{lll}
\sin\alpha=\dfrac{a}{c}, & a=c\sin\alpha, & c=\dfrac{a}{\sin\alpha}, \\[2ex]
\cos\alpha=\dfrac{b}{c}, & b=c\cos\alpha, & c=\dfrac{b}{\cos\alpha}, \\[2ex]
\tan\alpha=\dfrac{a}{b}, & a=b\tan\alpha, & b=\dfrac{a}{\tan\alpha}, \\[2ex]
\alpha+\beta=90^\circ, & \alpha=90^\circ-\beta, & \beta=90^\circ-\alpha.
\end{array}\right\} \qquad (21)$$

The cotangent, secant, and cosecant formulas are not needed often here because they express in reciprocal form the relations expressed in (21). Similarly, from (20),

$$\left.\begin{array}{lll}
\sin\beta=\dfrac{b}{c}, & b=c\sin\beta, & c=\dfrac{b}{\sin\beta}, \\[2ex]
\cos\beta=\dfrac{a}{c}, & a=c\cos\beta, & c=\dfrac{a}{\cos\beta}, \\[2ex]
\tan\beta=\dfrac{b}{a}, & b=a\tan\beta, & a=\dfrac{b}{\tan\beta}, \\[2ex]
\alpha+\beta=90^\circ, & \alpha=90^\circ-\beta, & \beta=90^\circ-\alpha.
\end{array}\right\} \qquad (22)$$

Formulas (21), (22) are called the "solution formulas" of the plane right triangle. We shall develop corresponding formulas for spherical right triangles in a later chapter.

9. *Formulas of Oblique Plane Triangles.*—An *oblique* plane triangle is defined as a plane triangle which has no right angle. Such a triangle cannot be fitted on the coordinate reference axes like a right triangle and therefore the solution formulas of the preceding article do not apply. Corresponding solution formulas for oblique plane triangles are derived in books on plane trigonometry such as those named at the end of this chapter, and the formulas will be listed here for reference but will not be proved or explained.

An oblique plane triangle may have any shape and its interior angles may be acute or obtuse but it cannot have more than one obtuse angle. The lettering shown in Fig. 10(*a*), (*b*) below should always be used. Fig. 10(*c*) shows an oblique triangle with its *circumscribed* and *inscribed circles* (sometimes called the *circumcircle*, or *out-circle*, and *in-circle*). The centers and radii of these circles are called the *circumcenter*, or *out-center*, and *in-center*, and the *out-radius* and *in-radius*, respectively.

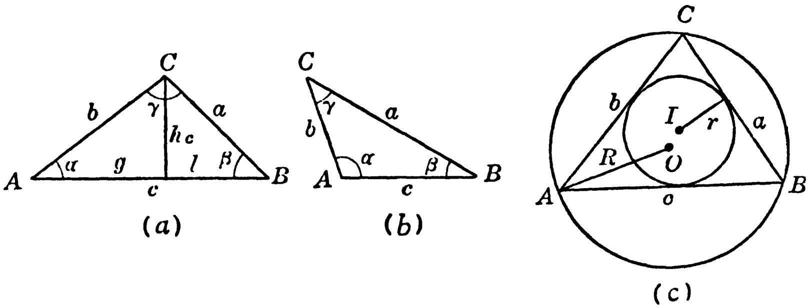


FIG. 10.

Side $\overline{AB}=c$ is generally taken as base of the triangle, and the following notation is used:

A, B, C are vertices of oblique plane triangle.
α, β, γ = interior angles of oblique plane triangle.
a, b, c = lengths of sides of oblique plane triangle.
$s=\frac{1}{2}(a+b+c)$ = half-sum of sides.
K = area of triangle in square units corresponding to a, b, c.
O, I are out-center and in-center.
R = radius of out-circle, in units of a, b, c.
r = radius of in-circle, in same units.
h_a, h_b, h_c = altitudes on sides a, b, c.
g = greater segment of base divided by altitude.
l = lesser segment of base.
$\alpha+\beta+\gamma=180°$ in every oblique plane triangle.

The most frequently useful formulas of the oblique triangle are the following:

$$\text{Sine Law:}\quad \frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}. \tag{23}$$

$$\text{Cosine Law:}\left\{\begin{aligned} a^2&=b^2+c^2-2bc\cos\alpha,\\ b^2&=c^2+a^2-2ca\cos\beta,\\ c^2&=a^2+b^2-2ab\cos\gamma.\end{aligned}\right\} \tag{24}$$

(In each case involving both sides and angles there are three formulas, as in (24), but only one will be given below in each case; the other two can be written out by interchanging sides and angles in (25) to (37).)

$$\text{Tangent Law:}\quad \frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}. \tag{25}$$

Half-angle Formulas:

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{26}$$

$$\cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}. \tag{27}$$

$$\tan\frac{\alpha}{2}=\frac{r}{s-a}. \tag{28}$$

Angle Formulas:

$$\sin\alpha=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \tag{29}$$

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}. \tag{30}$$

$$\tan\alpha=\frac{2r(s-a)}{(s-a)^2-r^2}. \tag{31}$$

$$\text{Segment Formulas:}\left\{\begin{array}{l} g+l=c, \quad g-l=\dfrac{(b+a)(b-a)}{c}; \\ \cos\alpha=\dfrac{g}{b}, \quad \cos\beta=\dfrac{l}{a}. \end{array}\right\} \tag{32}$$

$$\text{Mollweide Equations:}\left\{\begin{array}{l} \dfrac{a+b}{c}=\dfrac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}, \\ \dfrac{a-b}{c}=\dfrac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\gamma}. \end{array}\right\} \tag{33}$$

$$h_c=a\sin\beta=b\sin\alpha=\frac{2rs}{c}=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \tag{34}$$

$$K=\frac{ch_c}{2}=\tfrac{1}{2}bc\sin\alpha=\frac{a^2\sin\beta\sin\gamma}{2\sin(\beta+\gamma)}. \tag{35}$$

$$K=\frac{a^2\sin\beta}{2\sin\alpha}\sin(\alpha+\beta)=\frac{b^2\sin\alpha}{2\sin\beta}\sin(\alpha+\beta). \tag{36}$$

$$K=\frac{b^2}{2}\sin\alpha\left[\cos\alpha+\sqrt{\left(\frac{a}{b}\right)^2-\sin^2\alpha}\right]. \tag{37}$$

There is only one of each of the following formulas:

$$K=\frac{abc}{4R}=rs=\sqrt{s(s-a)(s-b)(s-c)}. \tag{38}$$

$$R=\frac{abc}{4K}=\frac{s}{4}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}. \tag{39}$$

$$r=\sqrt{\frac{1}{s}(s-a)(s-b)(s-c)}=s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}. \tag{40}$$

$$r\cdot R=\frac{abc}{4s}, \quad \frac{r}{R}=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}. \tag{41}$$

$$R^2=\frac{abc}{16s}\csc\frac{\alpha}{2}\csc\frac{\beta}{2}\csc\frac{\gamma}{2}, \quad r^2=\frac{abc}{s}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}. \tag{42}$$

10. *Formulas of the Circle.*—The geometry and trigonometry of the circle is sufficiently extensive to fill a large book but the following formulas are of frequent use. The notation is indicated in Fig. 11.

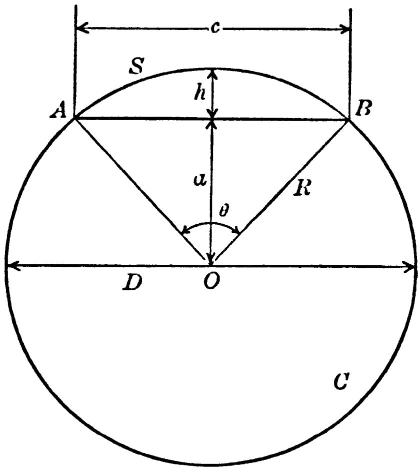


FIG. 11.

R = radius, K = area,
D = diameter, c = chord AB,
C = circumference, s = arc $\overset{\frown}{AB}$,
h = rise of arc $\overset{\frown}{AB}$,
θ = central $\angle AOB$ in radians,
S = area of segment AB,
u = area of sector AOB,
n = no. sides of inscribed regular polygon of side c, or of circumscribed regular polygon,
a = apothem of inscribed regular polygon,
p = perimeter of inscribed regular polygon,
P = perimeter of circumscribed regular polygon,
A_c = area of circumscribed regular polygon,
A_i = area of inscribed regular polygon.

Then $C=\pi D=2\pi R;\quad K=\pi R^2=\frac{1}{4}\pi D^2.$ (43)

$$s=R\theta=\tfrac{1}{2}D\theta=D\cos^{-1}\left(\frac{a}{R}\right). \tag{44}$$

$$\theta=\frac{s}{R}=2\sin^{-1}\left(\frac{c}{D}\right)=2\cos^{-1}\left(\frac{a}{R}\right)=2\tan^{-1}\left(\frac{c}{2a}\right) \tag{45}$$

$$a=\sqrt{R^2-\left(\frac{c}{2}\right)^2}=R\cos\frac{\theta}{2}=\frac{c}{2}\cot\frac{\theta}{2};\quad h=R-a. \tag{46}$$

$$c=2\sqrt{R^2-a^2}=D\sin\frac{\theta}{2}=2a\tan\frac{\theta}{2}. \tag{47}$$

$$S=\tfrac{1}{2}R^2(\theta-\sin\theta);\quad u=\tfrac{1}{2}Rs=\tfrac{1}{2}R^2\theta. \tag{48}$$

$$S=R^2\cos^{-1}\left(\frac{R-h}{R}\right)-(R-h)\sqrt{h(D-h)}. \tag{49}$$

$$A_i=\frac{n}{2}R^2\sin\left(\frac{2\pi}{n}\right)=\frac{n}{4}c^2\cot\left(\frac{\pi}{n}\right). \tag{50}$$

$$A_c=nR^2\tan\left(\frac{\pi}{n}\right)=\frac{n}{4}l^2\cot\left(\frac{\pi}{n}\right);\quad (l=\text{side}). \tag{51}$$

$$P=Dn\tan\left(\frac{\pi}{n}\right);\quad p=Dn\sin\left(\frac{\pi}{n}\right). \tag{52}$$

In these formulas the notation $\sin^{-1}$ is to be read "angle whose sine is," $\cos^{-1}$ means "angle whose cosine is," etc.

Plane, solid, and spherical geometry and plane and analytical trigonometry are treated in detail in the following books:

Wentworth, "Plane and Solid Geometry."
Phillips and Fisher, "Elements of Geometry."
Thompson, "Trigonometry for the Practical Man" (elementary).
Thompson and Cowles, "Text Book of Trigonometry" (advanced).

CHAPTER II

The Sphere and Spherical Triangles

11. *Introduction.*—It has been pointed out in Art. 3 that a knowledge of algebra, plane and solid geometry, and plane trigonometry is required for the study of spherical trigonometry. Chapter I presents a summary of certain parts of plane geometry pertaining to angles and angle measure, and a summary of certain parts of plane trigonometry. This chapter will present a summary of certain parts of solid geometry with particular reference to the sphere. For complete discussions and proofs reference should be made to books on geometry such as those mentioned at the end of Chapter I.

In this chapter all that pertains to the numerical measure of angles and to values and properties of the trigonometric or circular functions of angles, as set forth in Chapter I, will still apply, and tables of the natural and logarithmic functions will be used in this and following chapters as in plane trigonometry.

12. *Lines and Planes in Space.*—A *plane surface*, or simply a *plane*, is a surface such that a straight line passing through any two points of the surface lies entirely in the surface. If the flat surface of a smooth mirror or table top can be pictured or imagined as abstracted from the table, without thickness or body, we get a physical notion of a plane. All the discussions of Chapter I apply to lines, angles, and figures in a plane.

Consider now an isolated straight line in space. A plane may pass through this line and can turn about it as an axis, as illustrated by a piece of cardboard (which actually has thickness, however) held at two fixed points on its edges and turned about these

points. The cardboard (plane) then rotates about the straight line joining these two points. If now a third point in the cardboard (plane), outside the fixed line, be also fixed in position the plane cannot be turned. Thus the plane is fixed in position by the line and the point outside the line.

But *any* two points in the fixed line locate and determine it. These two, and the third point not in the line, therefore, also fix the plane in space. Similarly one plane can contain two intersecting or parallel lines, and while it can turn about either one of these two lines, if it must pass also through the other fixed line then it cannot turn but must remain in one fixed position.

The results of the last two paragraphs may be summarized as follows:

I. *A plane in space is determined by*

(*a*) *a straight line and a point outside the line;*
(*b*) *three points not in the same straight line;*
(*c*) *two intersecting straight lines; or*
(*d*) *two parallel straight lines.*

Since a curved line cannot lie wholly in a plane in every position into which the line may be turned but a straight line can (see definition of *plane* above), then a plane can turn about a straight line into any position, and

II. *The intersection of two planes is a straight line.*

This fact is illustrated below in Fig. 12, in which the two planes $ABCD$ and $ABEF$ intersect in the straight line AB. (Strictly speaking, $ABCD$ and $ABEF$ represent only a portion of each of two planes, and the planes may extend indefinitely beyond the boundaries AB, BE, BC, CD, etc.)

If a straight line pierces a plane at a point and is perpendicular to every line in the plane which passes through that point the line and the plane are said to be *perpendicular*, each to the other.

But since there can be only one line from a point outside perpendicular to each of these lines in the plane (Art. 4), then

III. *At a point in a plane or from a point outside the plane there can be only one straight line perpendicular to the plane;* and therefore also,

IV. *Through a point outside a line there can pass only one plane perpendicular to the line.*

A *plane angle* has been defined in Art. 4 as the figure formed by two intersecting straight lines, and the *measure* of such an angle is the amount of this opening or difference in direction between the lines. Similarly two intersecting planes may have different directions and we define:

A *dihedral angle* is the figure formed by two planes which meet in a line, and the measure of the dihedral angle is the amount of opening or difference in direction between the planes.

The two planes are called the *faces* of the dihedral angle, and their line of intersection is its *edge.*

The *measure* of a dihedral angle (in degrees or radians) is the same as the plane angle formed by two lines, one in each face, which are perpendicular to the edge at the same point. This is called the *plane angle of the dihedral angle.*

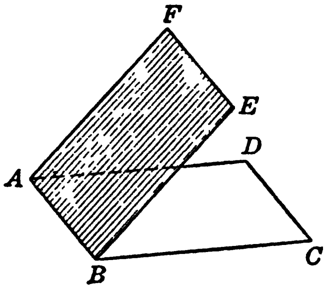


FIG. 12.

Fig. 12 represents a dihedral angle formed by the two intersecting planes $ABCD$ and $ABEF$. These planes are its faces, and their line of intersection AB is its edge. If the lines BC, BE are thought of as being both perpendicular to AB at B then the number of degrees or radians in the $\angle CBE$ is the measure of the dihedral angle. Thus a dihedral angle may have any magnitude, as in Arts. 4, 5.

A dihedral angle is designated by naming its edge or the edge and the face planes. Thus, in Fig. 12, dihedral $\angle AB$ or dihedral $\angle(AB\text{-}AC\text{-}AE)$.

If *three* planes all pass through one point and intersect one another in pairs in three lines, as the sides of a box meeting at a corner, the three planes are said to form a *trihedral angle.* The three planes are the *faces* of the trihedral angle; the three lines in which they intersect are its *edges;* the point through which they all pass is its *vertex;* and the three plane angles formed by the inter-

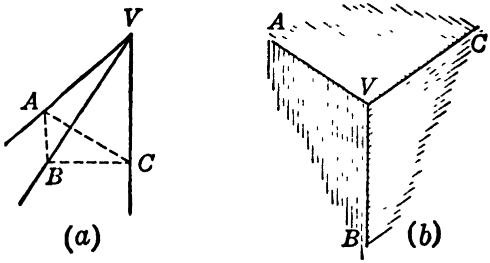


FIG. 13.

secting edges at the vertex are called the *face angles* of the trihedral angle.

Fig. 13 represents a trihedral angle. The triangular figures AVB, BVC, CVA represent portions of the three planes which meet at the point V. These three planes are the faces of the trihedral angle and V is its vertex. The lines AV, BV, CV are its edges, and the three plane $\angle s\ AVB$, BVC, CVA are its face angles. The trihedral angle is designated as trihedral $\angle V\text{-}ABC$.

Three *or more* planes which in the same way as above pass through one point and intersect in pairs in three or more straight lines are said to form a *polyhedral angle.* The polyhedral angle of any number of faces, edges, and face angles is described in the same manner as the angle of three faces (trihedral). A polyhedral angle is also called a *solid* angle in contradistinction from a *plane* angle.

If the plane angle of a dihedral angle is a right angle the face planes are perpendicular and it is called a *right dihedral angle.* Any two sides of a square or rectangular box meeting at an edge form a right dihedral angle.

If the face angles of a trihedral angle are plane right angles the trihedral angle is called a *right trihedral* angle. Thus the three sides of a square or rectangular box meeting at a corner form a right trihedral angle, as in Fig. 13(*b*).

If the edges AV, BV, CV in Fig. 13 spread farther apart (as the legs of a tripod) the face angles at V become larger and larger and the faces of the trihedral angle "flatten out." If this opening or flattening continues, the three planes will finally lie flat and coincide as one plane. The edges are then simply lines radiating from a point in a plane, and the sum of the three original face $\measuredangle AVB$, BVC, CVA is simply the sum of the plane angles about a point in a plane, which is 360 degrees or 2π radians.

From this we have the result that *the sum of the face angles must be less than 360 degrees* if the figure is to be a trihedral angle. Similarly, a polyhedral angle of any number of faces may be flattened out into a plane, when the polyhedral angle ceases to be a solid angle and the sum of its face angles becomes the sum of the plane angles about a point in a plane. Therefore

V. *The sum of the face angles of any polyhedral angle must be less than 360 degrees (2π radians).*

Lines and planes in space, and dihedral and polyhedral angles formed by them, possess many interesting properties and these are studied in solid geometry. Some of them are useful in the study of the sphere, which we next proceed to consider.

13. *The Sphere.*—A *sphere* is a solid, or portion of space, bounded by a curved surface, every point of which is at the same distance from one point within. This point is called the *center* of the sphere, and the distance from the center to any point in the

surface, or a line segment joining the center and a point in the surface, is called the *radius* of the sphere.

A line segment passing through the center of a sphere and terminating in the surface is called a *diameter* of the sphere. Obviously, a diameter is twice as long as the radius.

The intersection of the surface of a sphere and any plane is a circle. The intersection of a sphere and a plane through its center is a *great circle* of the sphere. All other circles formed by plane intersections with the sphere are called *small circles*.

The *axis* of a great circle is the diameter of the sphere which is perpendicular to the plane of the circle at its center. Each end of this diameter is called a *pole* of the great circle.

The radius, diameter, and circumference of a great circle are the radius (R), diameter (D), and circumference (C) of the sphere. The following relations are proved in geometry:

The circumference and area of a great circle of a sphere are

$$\left.\begin{aligned} C &= \pi D = 2\pi R, \\ A &= \tfrac{1}{2}CR = \pi R^2 = \tfrac{1}{4}\pi D^2. \end{aligned}\right\} \tag{1}$$

The surface area of a sphere is equal to the area of four of its great circles. Therefore

$$S = 2CR = 4\pi R^2 = \pi D^2. \tag{2}$$

The volume of a sphere is equal to two-thirds the volume of a circular cylinder whose diameter and altitude are each equal to the diameter of the sphere, or to one-third the volume of a cylinder having a circular base equal to the surface area of the sphere and an altitude equal to the radius of the sphere. Therefore

$$V = \tfrac{1}{3}RS = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3. \tag{3}$$

14. *Spherical Angles and Polygons.*—We now make use of the notion of the measure of an angle between two intersecting straight

lines to define the angle made by two intersecting *curved* lines: *The angle formed by two intersecting curved lines is measured by and equal to the angle formed by the tangents to the two curves at their point of intersection.*

Thus, in Fig. 14, the curved lines PB and PA intersect at P; PT is tangent to PA at P and PT' is tangent to PB at P; and the angle between PA and PB is equal to $\angle TPT'$.

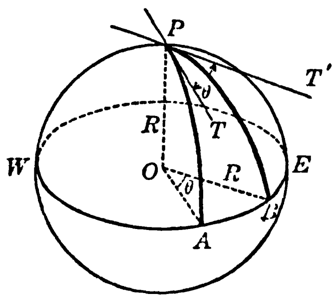


FIG. 14.

Also a tangent to a great circle of a sphere at a point is tangent to the sphere at that point and is perpendicular to the radius drawn to that point.

The angle formed by two intersecting arcs of great circles of a sphere (measured by the tangents) is called a *spherical angle.*

In Fig. 14 circle WAE represents a great circle on the surface of the sphere having its center at O.

The radius of the sphere and of the circle is $R = \bar{O}A = \bar{O}B = \bar{O}P$. OP is perpendicular to the plane of the great circle WAE and P is a pole of the circle.

$\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are arcs (quadrants) of great circles intersecting at P, and PT, PT' are tangents to these circles at P. The angle TPT' is the measure of the *spherical angle* formed by the arcs $\overset{\frown}{PA}$, $\overset{\frown}{PB}$. This spherical angle is denoted by $\angle APB$.

It is proved in geometry that

VI. *A spherical angle has the same measure as the dihedral angle formed by the planes of the great circles which form its sides.*

Thus in Fig. 14, spherical $\angle APB = \angle AOB = \theta$.

Also it has been seen in Art. 5 that $\angle AOB$ at the center of the

great circle WAE is measured by the arc $\overset{\frown}{AB}$ and its value in radians is $\angle AOB = \frac{\overset{\frown}{AB}}{OB} = \frac{\overset{\frown}{AB}}{R}$. Therefore also

VII. *A spherical angle is measured by the arc of the great circle having the vertex of the angle as pole and included between its sides (produced if necessary).*

In Fig. 14 the numerical value of the spherical $\angle APB$ in radians is therefore $\theta = \frac{\overset{\frown}{AB}}{R}$, where R is the radius of the sphere.

A *spherical right angle* is a spherical angle formed by two intersecting *perpendicular* arcs of great circles.

Thus, in Fig. 14, spherical angles $\angle PAB$, $\angle ABP$ are right angles, since the arcs $\overset{\frown}{PA}$, $\overset{\frown}{PB}$ are perpendicular to arc $\overset{\frown}{AB}$. The measure of each of these two angles is of course $\frac{\pi}{2}$ radians $= 90°$.

Spherical angles, like plane angles, may have any value, and similarly, any spherical angle greater than 2π radians $= 360°$ is geometrically equivalent to an angle less than 360° and having the same initial and terminal arcs as sides.

A *spherical polygon* is a closed figure formed by arcs of three or more great circles of the same sphere which intersect in pairs.

The *sides* of a spherical polygon are the great circular arcs which form it; the *angles* of the polygon are the spherical angles formed by intersecting pairs of its sides; and the *vertices* of the polygon are the points of intersection of its sides.

Spherical polygons may have any number of sides greater than two, and are named according to the number of sides, as are plane polygons. Thus there are *spherical triangles*, quadrilaterals, pentagons, etc.

An interior angle of a spherical polygon may have any value greater than zero and less than 360°.

A side of a spherical polygon, being an arc of the great circle of the sphere on which the polygon is constructed, may be measured in angular measure, i.e., degrees or radians of arc, or in linear units, as centimeters, inches, feet, etc., of length.

In Fig. 14, APB is a spherical polygon of three sides (triangle), the sides being the great circular arcs $\widehat{AB}$, $\widehat{BP}$, $\widehat{PA}$, and the vertices A, B, P. It is referred to as the *spherical* $\triangle ABP$.

The measure of the angles of $\triangle ABP$ have already been given. The *angular* measures of the *sides* are $\widehat{AP}=90°=\frac{\pi}{2}$, $\widehat{BP}=90°=\frac{\pi}{2}$ (quadrants), $\widehat{AB}=\angle\theta$. The *linear* measures of the sides are $\widehat{AP}=\widehat{BP}=\frac{\pi R}{2}$, $\widehat{AB}=R\theta$, by (1), Art. 5.

The remainder of this and the following chapters deal with the properties and solution of spherical triangles.

15. *Spherical Triangles.*—A spherical triangle has already been defined as a spherical polygon of three sides, which are arcs of great circles of a sphere, and three (interior) spherical angles. This definition may be separately and concisely stated as follows:

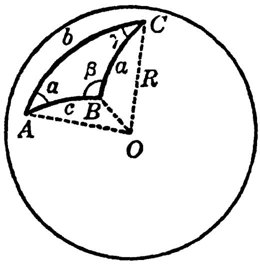


FIG. 15.

A *spherical triangle* is a portion of (or, a figure on) the surface of a sphere, bounded by three arcs of great circles of the sphere.

The bounding arcs are the *sides* of the triangle; the points of intersection of the sides are its *vertices;* and the interior spherical angles formed by pairs of the sides are the *angles of the triangle.*

In Fig. 15, ABC represents a spherical triangle, drawn on the surface of the sphere having the center O and the radius $\overline{OA}=\overline{OB}=\overline{OC}=R$.

A spherical triangle is easily visualized as the figure formed by the intersection of the surface of the sphere with the three faces of

a trihedral angle whose vertex is at the center of the sphere, as trihedral $\angle O\text{-}ABC$ in Fig. 15 with its vertex at the center O and face angles AOB, BOC, COA.

The Greek letters α, β, γ will be used to denote the radian or degree measure of the interior spherical angles of the triangle at A, B, C, respectively, and the italic letters a, b, c will denote the *angular* measure (degrees or radians) of the corresponding opposite sides which are the great circular arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$, respectively.

The *lengths* of the curved sides in linear measure will be denoted by the symbols $\bar{a}$, $\bar{b}$, $\bar{c}$. When a, b, c are expressed in radians then

$$\bar{a} = Ra, \quad \bar{b} = Rb, \quad \bar{c} = Rc, \tag{4}$$

and $\bar{a}$, $\bar{b}$, $\bar{c}$, R will be measured in the same linear units.

The sides and angles of a spherical triangle are called its *parts*.

A spherical triangle $A'B'C'$ whose vertices A', B', C' are the poles of the sides $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$, respectively of the spherical $\triangle ABC$, is said to be a *polar triangle* of $\triangle ABC$. Thus, in Fig. 16, A', B', C' are the poles of the three great circles of which the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$ are parts.

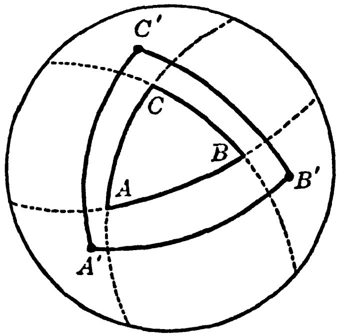


FIG. 16.

Since each side of a spherical triangle has two poles, the triangle has eight polar triangles; but there is only one $\triangle A'B'C'$ in which the vertices all lie towards the corresponding vertices A, B, C, respectively. This one of the eight polar triangles is called *the polar* of the spherical triangle ABC. Thus in Fig. 16 $\triangle A'B'C'$ is the polar of $\triangle ABC$.

The spherical $\triangle ABC$ is called the *primitive* of the polar $\triangle A'B'C'$.

It is proved in geometry that

VIII. *The angles and sides of a polar triangle are the supplements of the respective corresponding sides and angles of the primitive triangle, and conversely.*

A spherical triangle may have one, two, or three right angles.

A spherical triangle of three right angles is said to be *trirectangular*; one of two right angles is *birectangular*; and one having only one right angle is called a *spherical right triangle.*

As in plane triangles, the side opposite the right angle of a spherical right triangle is called the *hypotenuse.* The other two sides are called the *legs.*

A spherical triangle having no right angle is called an *oblique spherical triangle.*

A spherical triangle having two equal sides is said to be *isosceles.* In Fig. 14 the spherical $\triangle APB$ is isosceles, and also birectangular.

A *quadrantal* spherical triangle is one having one side equal to a quadrant of a great circle; a *biquadrantal* triangle has two sides equal to quadrants; and a *triquadrantal* triangle has three sides equal to quadrants.

A biquadrantal spherical triangle is of course isosceles; it is also birectangular. In Fig. 14, $\triangle APB$ is biquadrantal, the sides $\overset{\frown}{AP}$, $\overset{\frown}{BP}$ each being quadrants of great circles of the sphere.

A triquadrantal spherical triangle is also trirectangular, and of course equilateral. It bounds on *octant* of the sphere, and its area is one-eighth of the surface area of the sphere, i.e., $\frac{1}{2}\pi R^2$.

16. *Some Properties of Spherical Triangles.*—The following relations are proved in solid geometry. They are of the highest importance in spherical trigonometry, and attention paid to these

facts may often aid in avoiding errors in solutions or serve as checks on results:

IX. *The greater side of a spherical triangle is opposite the greater angle, and conversely. Equal sides are opposite equal angles.*

X. *Each side of a spherical triangle is less than the sum of the other two sides.*

XI. *The sum of the three sides of a spherical triangle is less than the circumference of its sphere;* or, in angular measure,

XII. *The sum of the sides of a spherical triangle is less than 360° (2π radians).*

XIII. *The sum of the interior angles of a spherical triangle is greater than 180° and less than 540° (π and 3π radians).*

The difference between the sum of the angles of a spherical triangle and 180° (π rad.) is called the *spherical excess* of the triangle.

It is proved in geometry that

XIV. *The area of a spherical triangle is to the surface area of the sphere as the spherical excess in degrees is to 720.*

If E is the spherical excess in degrees, S the surface area of the sphere in units of square measure, and K the area of the spherical triangle in the same units, then VIII is $K : S :: E : 720$.

$$\therefore \quad K = \frac{ES}{720}. \tag{5}$$

In terms of the radius and diameter of the sphere, $S = 4\pi^{D2} = \pi D^2$. Hence

$$K = \left(\frac{\pi R^2}{180}\right) E = \left(\frac{\pi D^2}{720}\right) E, \tag{6a}$$

when E is expressed in degrees.

If E is expressed in radians, $180° = \pi$ rad., and hence (6*a*) becomes

$$K = R^2E = \tfrac{1}{4}D^2E. \tag{6b}$$

In the general spherical triangle any side or angle may be greater than 180° (π radians), but in such cases it is always possible in numerical work to substitute for the triangle another in which each part is less than 180°.

In the following chapters, therefore, we shall consider only triangles in which each part is less than 180°.

Since sides and angles of spherical triangles are both measured in angular measure, the trigonometric functions of either may be used. Thus a side or arc of one-sixth of the circumference of the sphere is $60° = \frac{\pi}{3}$ radians. If this is, say, side a of a spherical triangle then $\sin a = .8660$, $\cos a = .5000$, etc. If the radius is $R = 15$, then the *length* of the side a is $\bar{a} = Ra = 5\pi = 15.708$ units.

In the next chapter we develop relations between the sides and angles of spherical triangles.

CHAPTER III

FORMULAS OF SPHERICAL TRIANGLES

17. *The Fundamental Formulas of the Spherical Triangle.*—In Fig. 17, ABC represents a spherical triangle on a sphere with center O and radius $\overline{OA} = \overline{OB} = \overline{OC} = R$. The angular measures of the sides are $\overset{\frown}{AB} = c$, $\overset{\frown}{BC} = a$, $\overset{\frown}{CA} = b$, and the corresponding opposite angles are spherical $\angle BCA = \gamma$, $\angle CAB = \alpha$, $\angle ABC = \beta$. The sides

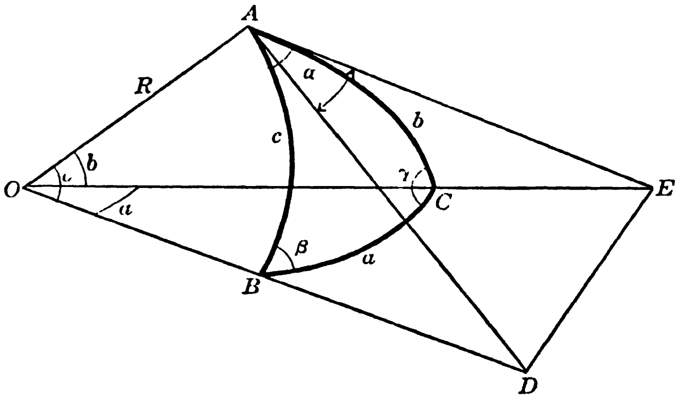


FIG. 17.

and angles may have any values between 0 and 180°, including 90°.

The sides $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ are arcs of great circles of the planes AOB, BOC, COA, respectively. These three planes through the center O may be visualized as cutting out of the sphere the trian-

gular "plug" or trihedral angle $O\text{-}ABC$ and intercepting or inclosing $\triangle ABC$ as a part of the surface of the sphere. The face angles AOB, BOC, COA of the trihedral angle are equal respectively to the angular measures of the sides c, a, b as indicated in Fig. 17.

AD is the tangent at A to the arc $\overset{\frown}{AB}$, meeting OB produced at D; and AE is the tangent at A to $\overset{\frown}{AC}$, meeting OC produced at E. Then by definition (Art. 14) plane $\angle DAE =$ spherical $\angle BAC = \alpha$. Similarly, spherical angles β and γ would be equal to the angles formed by drawing tangents to the sides at B and at C.

The $\angle s\ OAD$, OAE are right angles and the plane $\triangle s\ OAD$, OAE are right triangles. Therefore

$$\overline{OA}^2 = \overline{OD}^2 - \overline{AD}^2, \quad \overline{OA}^2 = \overline{OE}^2 - \overline{AE}^2. \tag{1}$$

In the plane triangles DOE, DAE, by the *cosine law* (24), Art. 9,

$$\overline{DE}^2 = \overline{OD}^2 + \overline{OE}^2 - 2\overline{OD}\cdot\overline{OE}\cos a,$$

$$\overline{DE}^2 = \overline{AD}^2 + \overline{AE}^2 - 2\overline{AD}\cdot\overline{AE}\cos\alpha.$$

Equating these two expressions for $\overline{DE}^2$, and transposing,

$$\begin{aligned} 2\overline{OD}\cdot\overline{OE}\cos a &= (\overline{OD}^2 - \overline{AD}^2) + (\overline{OE}^2 - \overline{AE}^2) + 2\overline{AD}\cdot\overline{AE}\cos\alpha \\ &= 2\overline{OA}^2 + 2\overline{AD}\cdot\overline{AE}\cos\alpha, \quad \text{by (1)}. \end{aligned}$$

$$\therefore \cos a = \frac{\overline{OA}\cdot\overline{OA}}{\overline{OE}\cdot\overline{OD}} + \frac{\overline{AE}\cdot\overline{AD}}{\overline{OE}\cdot\overline{OD}}\cos\alpha, \tag{2}$$

on dividing by $2\overline{OD}\cdot\overline{OE}$.

But in right triangles OAD, OAE

$$\frac{\overline{OA}}{\overline{OE}} = \cos b, \quad \frac{\overline{OA}}{\overline{OD}} = \cos c,$$

$$\frac{\overline{AE}}{\overline{OE}} = \sin b, \quad \frac{\overline{AD}}{\overline{OD}} = \sin c.$$

Substituting these relations in (2),

$$\cos a=\cos b\cos c+\sin b\sin c\cos\alpha.$$

Similarly,
$$\left.\begin{aligned}\cos a&=\cos b\cos c+\sin b\sin c\cos\alpha.\\ \cos b&=\cos c\cos a+\sin c\sin a\cos\beta,\\ \cos c&=\cos a\cos b+\sin a\sin b\cos\gamma.\end{aligned}\right\}\qquad(3)$$

The last two formulas are obtained by drawing the tangents to the sides at B, C, respectively, or by permutation of the parts of the triangle in the first formula. These derivations are left as exercises for the student. (See page 57.)

The equations (3) are called the *fundamental formulas* of the spherical triangle, and all other formulas of spherical triangles can be derived from them. We shall later make use of (3) in deriving the solution formulas for right and oblique spherical triangles.

18. *The Sine Law for Spherical Triangles.*—From the first of formulas (3) we have

$$\cos\alpha=\frac{\cos a-\cos b\cos c}{\sin b\sin c},$$

and also, for any angle, $\sin^2\alpha=1-\cos^2\alpha$.

$$\begin{aligned}\therefore\ \sin^2\alpha&=1-\left(\frac{\cos a-\cos b\cos c}{\sin b\sin c}\right)^2\\ &=\frac{\sin^2 b\sin^2 c-(\cos a-\cos b\cos c)^2}{\sin^2 b\sin^2 c}\\ &=\frac{(1-\cos^2 b)(1-\cos^2 c)-(\cos a-\cos b\cos c)^2}{\sin^2 b\sin^2 c}\\ &=\frac{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}{\sin^2 b\sin^2 c}\end{aligned}$$

$$\therefore\ \frac{\sin\alpha}{\sin a}=\frac{\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}}{\sin a\sin b\sin c}.$$

The right member of this equation contains all the sides a, b, c symmetrically and permutation of the sides does not change it. Permuting the sides and angles, therefore,

$$\frac{\sin\alpha}{\sin a}=\frac{\sin\beta}{\sin b}=\frac{\sin\gamma}{\sin c}. \tag{4}$$

This continued proportion expresses the *sine law* for spherical triangles. It is analogous to, but must not be confused with, the sine law for plane triangles, $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ (Art. 9).

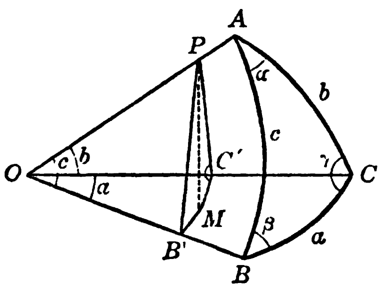


FIG. 18.

19. *Geometrical Proof of Sine Law.*—The sine law (4) may also be derived directly from a geometrical figure. The spherical triangle ABC of Fig. 17 is reproduced in Fig. 18. From any point P on OA draw $PM \perp$ plane OBC. Draw $MB' \perp OB$, $MC' \perp OC$, and join PB', PC'. Then plane $\triangle PMB'$, PMC' are right triangles; $\angle PB'M=\beta$, $\angle PC'M=\gamma$, and

$$\sin\beta=\frac{PM}{PB'},\quad \sin\gamma=\frac{PM}{PC'}. \tag{5}$$

Also, $PB' \perp OB$, $PC' \perp OC$. Hence plane $\triangle PB'O$, $PC'O$ are right triangles and

$$\sin b=\frac{PC'}{PO},\quad \sin c=\frac{PB'}{PO}. \tag{6}$$

From (5), and from (6),

$$\frac{\sin\beta}{\sin\gamma}=\frac{PC'}{PB'},\qquad \frac{\sin b}{\sin c}=\frac{PC'}{PB'}.$$

$$\therefore\ \frac{\sin\beta}{\sin\gamma}=\frac{\sin b}{\sin c},\quad \text{or}\quad \frac{\sin\beta}{\sin b}=\frac{\sin\gamma}{\sin c}.$$

By drawing the perpendicular to plane OAB from a point on OC we find similarly $\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b}$, and hence

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}. \tag{4}$$

20. *Formulas for a Side and the Three Angles.*—According to VIII, Art. 15, we have for the polar triangle of $\triangle ABC$ (Fig. 16)

$$\left.\begin{array}{ll} \alpha' = 180° - a, & a' = 180° - \alpha, \\ \beta' = 180° - b, & b' = 180° - \beta, \\ \gamma' = 180° - c, & c' = 180° - \gamma, \end{array}\right\} \tag{7}$$

and in the polar $A'B'C'$, according to the first of the fundamental formulas (3),

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \alpha'.$$

Replacing α', a', b', c' in this formula by the values in (7) and remembering that $\cos (180° - \alpha) = -\cos \alpha$, $\cos (180° - a) = -\cos a$, $\sin (180° - a) = \sin a$, etc., we have

$$-\cos \alpha = (-\cos \beta)(-\cos \gamma) + \sin \beta \sin \gamma \, (-\cos a),$$

or

$$\left.\begin{array}{l} \cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a. \\ \text{Similarly, } \cos \beta = -\cos \gamma \cos \alpha + \sin \gamma \sin \alpha \cos b, \\ \cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c. \end{array}\right\} \tag{8}$$

Each of these three formulas contains one side and all three angles of the spherical triangle.

21. *Formulas for the Half-angles of a Spherical Triangle.*—From the first fundamental formula (3),

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{9}$$

$$\therefore\ 2\sin^2\frac{\alpha}{2}=1-\cos\alpha=1-\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{(\sin b\sin c+\cos b\cos c)-\cos a}{\sin b\sin c}.$$

$$\therefore\ 2\sin^2\frac{\alpha}{2}=\frac{\cos(b-c)-\cos a}{\sin b\sin c}.$$

By (18), Art. 7, $\cos(b-c)-\cos a=2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)$.

$$\therefore\ \sin^2\frac{\alpha}{2}=\frac{\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b\sin c}. \tag{10}$$

Now let

then

$$\left.\begin{aligned} a+b+c&=2s;\\ -a+b+c&=2(s-a),\\ a-b+c&=2(s-b),\\ a+b-c&=2(s-c).\end{aligned}\right\} \tag{11}$$

Substituting these in (10) we have

Similarly,

$$\left.\begin{aligned} \sin^2\frac{\alpha}{2}&=\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}.\\ \sin^2\frac{\beta}{2}&=\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a},\\ \sin^2\frac{\gamma}{2}&=\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}.\end{aligned}\right\} \tag{12}$$

Using (9) again we have

$$2\cos^2\frac{\alpha}{2}=1+\cos\alpha=1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos a-(\cos b\cos c-\sin b\sin c)}{\sin b\sin c}.$$

$$\therefore\ 2\cos^2\frac{\alpha}{2}=\frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

Reducing this by means of the relation (17), Art. 7,

$$\cos a-\cos(b+c)=-2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(a-b-c)$$
$$=\ 2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(-a+b+c),$$

and again using (11), we obtain

Similarly,

$$\left.\begin{aligned}\cos^2\frac{\alpha}{2}&=\frac{\sin s\sin(s-a)}{\sin b\sin c}.\\ \cos^2\frac{\beta}{2}&=\frac{\sin s\sin(s-b)}{\sin c\sin a},\\ \cos^2\frac{\gamma}{2}&=\frac{\sin s\sin(s-c)}{\sin a\sin b}.\end{aligned}\right\}\tag{13}$$

Dividing the first of formulas (12) by the first of (13), we obtain

$$\tan^2\frac{\alpha}{2}=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}=\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin^2(s-a)\sin s}.$$

Now let

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}\tag{14}$$

and take the square root of the last equation.

We get

Similarly,

$$\left.\begin{aligned}\tan\frac{\alpha}{2}&=\frac{\tan r}{\sin(s-a)}.\\ \tan\frac{\beta}{2}&=\frac{\tan r}{\sin(s-b)},\\ \tan\frac{\gamma}{2}&=\frac{\tan r}{\sin(s-c)}.\end{aligned}\right\}\tag{15}$$

Formulas (12), (13), (15) express each angle of any spherical triangle in terms of the three sides. They are analogous to the plane triangle formulas (26), (27), (28) of Art. 9.

22. *Formulas for the Half-sides.*—From the first of formulas (8),

$$\cos a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

Forming the expressions for $1-\cos a$ and $1+\cos a$, as in the preceding article for $\cos\alpha$, reducing in a similar manner, and putting

$$S = \tfrac{1}{2}(\alpha+\beta+\gamma), \tag{16}$$

we get

$$\left.\begin{aligned} \sin^2\frac{a}{2} &= -\frac{\cos S\cos(S-\alpha)}{\sin\beta\sin\gamma}, \\ \sin^2\frac{b}{2} &= -\frac{\cos S\cos(S-\beta)}{\sin\gamma\sin\alpha}, \\ \sin^2\frac{c}{2} &= -\frac{\cos S\cos(S-\gamma)}{\sin\alpha\sin\beta}; \end{aligned}\right\} \tag{17}$$

and

$$\left.\begin{aligned} \cos^2\frac{a}{2} &= \frac{\cos(S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}, \\ \cos^2\frac{b}{2} &= \frac{\cos(S-\gamma)\cos(S-\alpha)}{\sin\gamma\sin\alpha}, \\ \cos^2\frac{c}{2} &= \frac{\cos(S-\alpha)\cos(S-\beta)}{\sin\alpha\sin\beta}. \end{aligned}\right\} \tag{18}$$

Dividing the first of (17) by the first of (18) and putting

$$\tan R = \sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}, \tag{19}$$

we get

$$\tan\frac{a}{2} = \tan R\cos(S-\alpha).$$

Similarly,

$$\left.\begin{aligned} \tan\frac{a}{2} &= \tan R\cos(S-\alpha). \\ \tan\frac{b}{2} &= \tan R\cos(S-\beta), \\ \tan\frac{c}{2} &= \tan R\cos(S-\gamma). \end{aligned}\right\} \tag{20}$$

According to XII, Art. 16, the sum $\alpha+\beta+\gamma$ is between 180° and 540°. Hence $S=\frac{1}{2}(\alpha+\beta+\gamma)$ is between 90° and 270°. In (19), therefore, $\cos S$ is always negative and hence $-\cos S$ is always positive.

23. *Delambre's Equations.*—Multiplying the first of equations (12) by the second of (13), and taking the square root of the product equation, we get

$$\sin\frac{\alpha}{2}\cos\frac{\beta}{2}=\frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}},$$

or, comparing with the third of (13),

$$\sin\frac{\alpha}{2}\cos\frac{\beta}{2}=\frac{\sin(s-b)}{\sin c}\cos\frac{\gamma}{2}.$$

Similarly,

$$\left.\begin{aligned}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}=\frac{\sin(s-a)}{\sin c}\cos\frac{\gamma}{2};\end{aligned}\right\}\quad(21)$$

and

$$\left.\begin{aligned}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}&=\frac{\sin(s-c)}{\sin c}\sin\frac{\gamma}{2},\\ \cos\frac{\alpha}{2}\cos\frac{\beta}{2}&=\frac{\sin s}{\sin c}\sin\frac{\gamma}{2}\end{aligned}\right\}\quad(22)$$

Adding the two equations (21) and taking out $\cos\frac{\gamma}{2}$ as a factor on the right,

$$\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=\cos\frac{\gamma}{2}\cdot\frac{\sin(s-b)+\sin(s-a)}{\sin c},$$

or, $$\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\sin\tfrac{1}{2}(\alpha+\beta)=\cos\frac{\gamma}{2}\cdot\frac{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{\sin c}$$

$$=\cos\frac{\gamma}{2}\cdot\frac{2\sin\frac{c}{2}\cos\frac{1}{2}(a-b)}{2\sin\frac{c}{2}\cos\frac{c}{2}}$$

$$\therefore \quad \sin \tfrac{1}{2}(\alpha+\beta) = \cos \frac{\gamma}{2} \cdot \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{c}{2}}.$$

$$\therefore \quad \cos \frac{c}{2} \sin \tfrac{1}{2}(\alpha+\beta) = \cos \frac{\gamma}{2} \cos \tfrac{1}{2}(a-b). \qquad (23a)$$

Similarly, by subtracting the second of equations (21) from the first, and reducing, we obtain

$$\sin \frac{c}{2} \sin \tfrac{1}{2}(\alpha-\beta) = \cos \frac{\gamma}{2} \sin \tfrac{1}{2}(a-b). \qquad (23b)$$

Similarly, by adding the two equations (22), and by subtracting the second of (22) from the first, and reducing in each case as above we get, respectively,

$$\sin \frac{c}{2} \cos \tfrac{1}{2}(\alpha-\beta) = \sin \frac{\gamma}{2} \sin \tfrac{1}{2}(a+b), \qquad (23c)$$

$$\cos \frac{c}{2} \cos \tfrac{1}{2}(\alpha+\beta) = \sin \frac{\gamma}{2} \cos \tfrac{1}{2}(a+b). \qquad (23d)$$

Writing the formulas (23) together in the order (a), (d), (b), (c), we have finally

$$\left.\begin{aligned}
\cos \frac{c}{2} \sin \tfrac{1}{2}(\alpha+\beta) &= \cos \frac{\gamma}{2} \cos \tfrac{1}{2}(a-b), \\
\cos \frac{c}{2} \cos \tfrac{1}{2}(\alpha+\beta) &= \sin \frac{\gamma}{2} \cos \tfrac{1}{2}(a+b), \\
\sin \frac{c}{2} \sin \tfrac{1}{2}(\alpha-\beta) &= \cos \frac{\gamma}{2} \sin \tfrac{1}{2}(a-b), \\
\sin \frac{c}{2} \cos \tfrac{1}{2}(\alpha-\beta) &= \sin \frac{\gamma}{2} \sin \tfrac{1}{2}(a+b).
\end{aligned}\right\} \qquad (23)$$

These are known as *Delambre's Equations*, from the French mathematician who discovered them in 1807 and first published them in 1809. They are also sometimes improperly called Gauss's Equations.

24. *Napier's Ratios.*—Dividing the third of the group of equations (23) by the first, we get

$$\tan\frac{c}{2}\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}=\tan\tfrac{1}{2}(a-b).$$

Transposing, Similarly,

$$\left.\begin{aligned}\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}&=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}.\\ \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}&=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)},\end{aligned}\right\}\qquad(24a)$$

from the quotient of the fourth and second of (23).

Dividing the third of (23) by the fourth, and the first by the second, and reducing as above, we get, respectively,

$$\left.\begin{aligned}\frac{\tan\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}\gamma}&=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)},\\ \frac{\tan\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}\gamma}&=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\end{aligned}\right\}\qquad(24b)$$

The four proportions (24) are known as *Napier's Ratios*. They were first stated by the Scottish Baron John Napier (1550–1617), the inventor of logarithms, who also made other contributions to trigonometry.

25. *Additional Formulas for Sides and Angles.*—By transposing the second fundamental formula (3),

$$\sin c\sin a\cos\beta=\cos b-\cos c\cos a,$$

and by substituting in this the value of cos a from the first of (3),

$$\sin c \sin a \cos \beta = \cos b - \cos c(\cos b \cos c + \sin b \sin c \cos \alpha)$$
$$= \cos b - \cos b \cos^2 c - \sin b \sin c \cos c \cos \alpha$$
$$= \cos b(1 - \cos^2 c) - \sin b \sin c \cos c \cos \alpha$$
$$= \cos b \sin^2 c - \sin b \sin c \cos c \cos \alpha.$$

$$\therefore \sin c \sin a \cos \beta = (\cos b \sin c - \sin b \cos c \cos \alpha) \sin c.$$

$$\therefore \sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha. \qquad (25)$$

Two other similar formulas are obtained by permuting the parts.

Applying VIII, Art. 15, to (25) and then omitting the prime marks for the polar triangle (from α', β', γ', a', b', c'), we get

$$\sin \alpha \cos b = \cos \beta \sin \gamma - \sin \beta \cos \gamma \cos a. \qquad (26)$$

We have from the first sine proportion of (4) $\sin a \sin \beta = \sin b \sin \alpha$; dividing (25) by this equation, we have

$$\cot \beta = \frac{\cot b \sin c - \cos c \sin \alpha}{\sin \alpha}.$$

$$\therefore \sin \alpha \cot \beta = \cot b \sin c - \cos c \sin \alpha. \qquad (27)$$

Transposing this,

$$\sin c \cot b = \sin \alpha \cot \beta + \cos c \sin \alpha,$$

and, on permuting the sides and angles of this equation in order, we get

$$\sin a \cot b = \cot \beta \sin \gamma + \cos a \cos \gamma. \qquad (28)$$

Two other similar formulas may be obtained from each of (27) and (28) by permuting the parts in the same way.

26. *Formulas for Spherical Excess.*—By definition, the spherical excess of a spherical triangle, measured in degrees, is

$$E = \alpha + \beta + \gamma - 180°.$$

When the three angles are given or have been computed, this re-

lation gives E directly, and the area of the triangle is then found by the formulas of Art. 16 when the radius of the sphere is known.

A formula which gives E in terms of the three sides is found as follows:

$$\tan\frac{E}{4}=\frac{\sin\frac{1}{4}E}{\cos\frac{1}{4}E}=\frac{\sin\frac{1}{4}(\alpha+\beta+\gamma-\pi)}{\cos\frac{1}{4}(\alpha+\beta+\gamma-\pi)}.$$

Multiplying numerator and denominator of the last fraction by $2\cos\frac{1}{4}(\alpha+\beta+\pi-\gamma)$,

$$\tan\frac{E}{4}=\frac{2\cos\frac{1}{4}[(\alpha+\beta)+(\pi-\gamma)]\sin\frac{1}{4}[(\alpha+\beta)-(\pi-\gamma)]}{2\cos\frac{1}{4}[(\alpha+\beta)+(\pi-\gamma)]\cos\frac{1}{4}[(\alpha+\beta)-(\pi-\gamma)]}$$

$$=\frac{\sin\frac{1}{2}(\alpha+\beta)-\sin\frac{1}{2}(\pi-\gamma)}{\cos\frac{1}{2}(\alpha+\beta)+\cos\frac{1}{2}(\pi-\gamma)},\qquad\text{by (17), (18), Art. 7.}$$

$$\therefore\quad\tan\frac{E}{4}=\frac{\sin\frac{1}{2}(\alpha+\beta)-\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}(\alpha+\beta)+\sin\frac{1}{2}\gamma}.\tag{29}$$

Now by the first two of Delambre's Equations (23),

$$\left.\begin{aligned}\sin\tfrac{1}{2}(\alpha+\beta)&=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cdot\cos\frac{\gamma}{2},\\[2ex]\cos\tfrac{1}{2}(\alpha+\beta)&=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\cdot\sin\frac{\gamma}{2}.\end{aligned}\right\}\tag{30}$$

Substituting these in (29),

$$\tan\frac{E}{4}=\frac{\dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cdot\cos\dfrac{\gamma}{2}-\cos\dfrac{\gamma}{2}}{\dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\cdot\sin\dfrac{\gamma}{2}+\sin\dfrac{\gamma}{2}}$$

$$=\frac{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}\cdot\frac{\cos\frac{\gamma}{2}}{\sin\frac{\gamma}{2}}$$

$$=\frac{2\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(-a+b+c)}{2\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\cdot\frac{\cos\frac{\gamma}{2}}{\sin\frac{\gamma}{2}},\qquad\text{by (18), Art. 7.}$$

$$=\frac{\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(-a+b+c)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\cot\frac{\gamma}{2}.$$

$$\therefore\ \tan\frac{E}{4}=\frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\cot\frac{\gamma}{2}\tag{31}$$

Now by inverting the third of the formulas (15) and using (14) we find

$$\cot\frac{\gamma}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}},$$

and this, in (31), gives

$$\tan\frac{E}{4}=\frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}},$$

$$=\frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\times$$

$$\sqrt{\frac{2\sin\frac{1}{2}s\cos\frac{1}{2}s\cdot 2\sin\frac{1}{2}(s-c)\cos\frac{1}{2}(s-c)}{2\sin\frac{1}{2}(s-a)\cos\frac{1}{2}(s-a)\cdot 2\sin\frac{1}{2}(s-b)\cos\frac{1}{2}(s-b)}}.$$

$$\therefore\ \tan^2\frac{E}{4}=\frac{\sin^2\frac{1}{2}(s-b)\sin^2\frac{1}{2}(s-a)}{\cos^2\frac{1}{2}s\cos^2\frac{1}{2}(s-c)}\times$$

$$\frac{\sin\frac{1}{2}s\cos\frac{1}{2}s\sin\frac{1}{2}(s-c)\cos\frac{1}{2}(s-c)}{\sin\frac{1}{2}(s-a)\cos\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\cos\frac{1}{2}(s-b)}$$

$$\therefore \tan^2\frac{E}{4}=\frac{\sin\frac{1}{2}s\cdot\sin\frac{1}{2}(s-a)\cdot\sin\frac{1}{2}(s-b)\cdot\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}s\cdot\cos\frac{1}{2}(s-a)\cdot\cos\frac{1}{2}(s-b)\cdot\cos\frac{1}{2}(s-c)}$$

$$=\tan\tfrac{1}{2}s\cdot\tan\tfrac{1}{2}(s-a)\cdot\tan\tfrac{1}{2}(s-b)\cdot\tan\tfrac{1}{2}(s-c)$$

$$\therefore \tan\frac{E}{4}=\sqrt{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)}. \quad (32)$$

This formula gives the spherical excess E in terms of the sides a, b, c, where $s=\frac{1}{2}(a+b+c)$. It is known as *Lhuillier's Formula.*

Another convenient formula which gives E in terms of any *two sides and their included angle* is found as follows:

$$\tan\frac{E}{2}=\frac{\sin\frac{E}{2}}{\cos\frac{E}{2}}=\frac{\sin\frac{1}{2}(\alpha+\beta+\gamma-\pi)}{\cos\frac{1}{2}(\alpha+\beta+\gamma-\pi)}=-\frac{\cos\frac{1}{2}(\alpha+\beta+\gamma)}{\sin\frac{1}{2}(\alpha+\beta+\gamma)}.$$

Now

$$-\cos\tfrac{1}{2}(\alpha+\beta+\gamma)=-\cos[\tfrac{1}{2}(\alpha+\beta)+\tfrac{1}{2}\gamma]$$

$$=-[\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}\gamma-\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}\gamma]$$

$$=\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}\gamma-\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}\gamma,$$

and $\sin\frac{1}{2}(\alpha+\beta+\gamma)=\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}\gamma+\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}\gamma.$

$$\therefore \tan\frac{E}{2}=\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}\gamma-\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}\gamma+\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}\gamma}.$$

Substituting (30) for $\sin\frac{1}{2}(\alpha+\beta)$ and $\cos\frac{1}{2}(\alpha+\beta)$, and reducing, this becomes

$$\tan\frac{E}{2}=\frac{\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}[\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(a-b)\cos^2\frac{\gamma}{2}+\cos\frac{1}{2}(a+b)\sin^2\frac{\gamma}{2}}.$$

But $\cos^2\frac{\gamma}{2}=\frac{1}{2}(1+\cos\gamma)$ and $\sin^2\frac{\gamma}{2}=\frac{1}{2}(1-\cos\gamma)$. Hence $\tan\frac{E}{2}$

$$=\frac{\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\left[2\sin\frac{a}{2}\sin\frac{b}{2}\right]}{\frac{1}{2}[\cos\frac{1}{2}(a-b)+\cos\frac{1}{2}(a+b)]+\frac{1}{2}[\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)]\cos\gamma}$$

$$=\frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin\gamma}{\left[\cos\frac{a}{2}\cos\frac{b}{2}\right]+\left[\sin\frac{a}{2}\sin\frac{b}{2}\right]\cos\gamma},\ \text{since } 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}=\sin\gamma.$$

Dividing numerator and denominator of this fraction by $\cos\frac{a}{2}\cos\frac{b}{2}$, we get finally,

$$\tan\frac{E}{2}=\frac{\tan\frac{a}{2}\tan\frac{b}{2}\sin\gamma}{1+\tan\frac{a}{2}\tan\frac{b}{2}\cos\gamma}, \tag{33}$$

which gives E in terms of the two sides a, b and their included angle γ. The same formula serves for b, c, α and for c, a, β on changing symbols.

On inverting (33), it becomes

$$\cot\frac{E}{2}=\frac{\cot\frac{a}{2}\cot\frac{b}{2}}{\sin\gamma}+\cot\gamma, \tag{34}$$

with corresponding formulas for b, c, α and c, a, β.

This completes a list of formulas sufficient for our purposes. In the general theory of spherical triangles many more such formulas are derived but those derived above include all that we will need for the numerical solution of any spherical triangle. These are used in the next two chapters.

Exercises

1. By means of a figure similar to Fig. 17, with tangents drawn at B, derive the second of formulas (3).

2. Similarly, with tangents at C, derive the third of formulas (3).

3. By the method of Art. 18, writing $\sin^2 \beta = 1 - \cos^2 \beta$, and solving the second of formulas (3) for $\cos \beta$, derive the expression for $\sin \beta / \sin b$.

4. Similarly derive the expression for $\sin \gamma / \sin c$.

5. By means of a figure similar to Fig. 18, with the perpendicular drawn to plane OAB from a point on OC, prove that $\sin \alpha / \sin a = \sin \beta / \sin b$.

6. By the method of Art. 20, applying the second of formulas (3) to the polar of ΔABC, derive the second of formulas (8).

7. Similarly derive the third of formulas (8).

8. By the method of Art. 21, using the second of formulas (3) and applying (11), derive the second of formulas (12).

9. Similarly derive the third of formulas (12).

10. By the method used in Art. 21 to derive the first of formulas (13), derive the second of formulas (13).

11. Similarly derive the third of formulas (13).

12. A small circle drawn on the surface of a sphere tangent to the sides of a spherical triangle is called the *inscribed* circle of the triangle. The arc of a great circle drawn from the pole of the inscribed circle to its circumference is called the *radius* of the inscribed circle. Prove that the radius (r) of the inscribed circle is given by formula (14).

13. Prove by the use of the first of formulas (15) that

$$\tan r = \sin a \sec \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$$

14. A small circle drawn on the surface of a sphere and passing through the vertices of a spherical triangle is called the *circumscribed* circle of the triangle, and the radius of the circle is defined as in Ex. 12. Prove that the radius (R) of the circumscribed circle is given by either of formulas (20) and hence by (19).

15. Prove by the use of the first of formulas (20) that

$$\tan R = \sin \alpha \sin \frac{a}{2} \sec \frac{b}{2} \sec \frac{c}{2}.$$

16. Show that in an equilateral spherical triangle $\tan R = 2 \tan r$.

17. By the method used in Art. 25 to derive formula (27), derive a similar formula for $\sin \beta \cot \gamma$.

18. Similarly, derive the corresponding expression for $\sin \gamma \cot \alpha$.

19. Write the formula for $\sin b \cot c$ corresponding to (28).

20. Similarly write the corresponding formula for $\sin c \cot a$.

CHAPTER IV

SOLUTION OF SPHERICAL RIGHT TRIANGLES

27. *Solutions of Spherical Right Triangles.*—As with plane triangles the *solution* of a spherical triangle is the operation of finding the values of the remaining unknown parts when certain of its parts are given. One of the six parts of a spherical right triangle being fixed and known (the right angle, 90°), five parts are subject to variation, and two more parts must be given in each case in order to find the remaining three parts.

Two of five parts may be given in six different ways, as follows:

Case I. The hypotenuse and an angle.
II. The hypotenuse and a leg.
III. The two angles (other than 90°).
IV. The two legs.
V. An angle and the adjacent leg.
VI. An angle and the opposite leg.

It will be noted that the spherical right triangle may be solved when the two angles (other than the right angle) are given. This is true because the triangle is completely determined, as any change of shape also entails a change of size of the triangle on a given sphere. This is not true of a triangle in a plane with straight sides.

As the trigonometric functions of the sides as well as of the angles are involved, the spherical triangle cannot be solved by means of the slide rule as plane triangles are solved, and as the formulas are somewhat more complicated than those of plane triangles, direct solutions by means of natural functions are not so simple and convenient. All computation is, therefore, to be done

with logarithms. In this book five figures are to be used and all angular values are to be expressed in degree measure to the nearest whole second.

Methods of laying out the work are given in the following articles and these should be followed in all work.

Solution formulas for spherical right triangles will now be developed from the general formulas of the preceding chapter, for all of the six cases given above.

28. *Solution Formulas for Right Triangles.*—We have from the fundamental formulas (3), Art. 17,

$$\cos c=\cos a \cos b+\sin a \sin b \cos \gamma. \tag{a}$$

From (8), Art. 20,
$$\begin{cases}\cos \alpha=-\cos \beta \cos \gamma+\sin \beta \sin \gamma \cos a, & (b)\\ \cos \gamma=-\cos \alpha \cos \beta+\sin \alpha \sin \beta \cos c. & (c)\end{cases}$$

From (4), Art. 18,
$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}. \tag{d}$$

From (25), Art. 25,
$$\sin a \cos \gamma=\cos c \sin b-\sin c \cos b \cos \alpha. \tag{e}$$

From (27), Art. 25,
$$\sin \gamma \cot \alpha=\cot a \sin b-\cos b \cos \gamma. \tag{f}$$

Unless otherwise specified we shall always let the right angle of a right triangle be at C, with side c as the hypotenuse and sides a, b as the legs. Then $\gamma=90°$, $\sin \gamma=1$, $\cos \gamma=0$, and the formulas (a) to (f) give the following formulas for spherical *right triangles.*

From (a), $$\cos c=\cos a \cos b. \tag{1}$$

From (c), $$\cos c=\cot \alpha \cot \beta. \tag{2}$$

From (b), $$\cos \alpha=\sin \beta \cos a. \tag{3}$$

Permuting (3), $$\cos \beta=\sin \alpha \cos b. \tag{4}$$

From (e), $$\cos \alpha=\tan b \cot c. \tag{5}$$

Permuting (5), $$\cos \beta=\tan a \cot c. \tag{6}$$

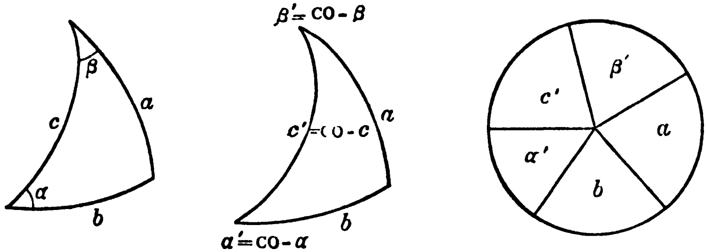
From (d), $\begin{cases} \sin a = \sin c \sin \alpha, & (7) \\ \sin b = \sin c \sin \beta. & (8) \end{cases}$

From (f), $\sin b = \tan a \cot \alpha.$ (9)

Permuting (9), $\sin a = \tan b \cot \beta.$ (10)

These ten formulas are sufficient for the solution of any spherical right triangle in which any two parts besides the right angle are given.

29. *Napier's Rules.*—All of the ten solution formulas given above may be easily written out by means of two simple rules devised by Baron Napier, the inventor of logarithms.



(a) (b) (c)

Fig. 19.

Omitting the right angle and taking the *complements* of the hypotenuse and the other angles, he called the five parts b, a, co-$\beta = \beta'$, co-$c = c'$, co-$\alpha = \alpha'$, the *circular parts* of the spherical right triangle (Fig. 19(a)), and marked the figure as in Fig. 19(b). The order of the "circular parts" of (b) is shown in Fig. 19(c).

If we consider any three of the five circular parts, either one will lie between the other two, being "adjacent" to both; or one will be "opposite" the other two, being separated from them by the two remaining parts.

Thus of the parts a, b, α', part b is between and "adjacent" to a, α'; of the parts b, c', β', b is "opposite" c' and β', being separated from them by a and α'; of the parts a, c', α', a is "opposite" c' and α', being separated from them by b and β'. One or the other of these relations will always be true for any three parts of the five (omitting the right angle).

Napier called that one of any three parts which is either adjacent to or opposite the other two, the *middle* part, and the other two the *adjacent* parts or *opposite* parts, respectively, as the case may be.

Napier's two *Rules of Circular Parts* are then:

A. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*
B. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

The rules may be easily remembered by the **i** in the words sine and middle, the **a** in tangent and adjacent, and the **o** in cosine and opposite.

As an example of the use of the rules, suppose in a right triangle solution a, b given and α, β, c to be found. To find α we require the formula containing a, b, α. Considering the circular parts a, b, co-$\alpha = \alpha'$, the circle diagram shows that b is the middle part and a, α' are adjacent parts. By Rule (A), $\sin b = \tan a \tan \alpha' = \tan a \cot \alpha$, which is solution formula (9), Art. 28.

To find β: considering a, b, β' the diagram shows a is the middle part and b, β' adjacent parts. By Rule (A), $\sin a = \tan b \tan \beta' = \tan b \cot \beta$, which is solution formula (10).

To find c: c' is the middle part and a, b opposite parts. By Rule (B), $\sin c' = \cos a \cos b$, or $\cos c = \cos a \cos b$, which is formula (1).

A check or verification formula for the solution may be found by considering the three required parts together α', β', c'. Here c'

is the middle part and α', β' adjacent. By Rule (A), $\sin c' = \tan \alpha' \tan \beta'$, or $\cos c = \cot \alpha \cot \beta$, formula (2).

Similarly the required formulas for any solution may be chosen from the list (1)–(10) or written out by Napier's Rules of circular parts.

30. *Quadrant in which a Computed Part Lies.*—As all the parts (angles and sides) of a spherical triangle are less than 180°, the sign of the function determines the value of a computed part, that is, the quadrant in which it lies, when the part is found from its cosine, tangent, or cotangent. If it is found from the sine function it may lie in either the first or second quadrant, as the sine function is positive in both those quadrants. (See chart, p. 13.)

Except in the particular case of a double solution ((E), below) the proper quadrant for a part (angle or side) found from its sine may always be determined by one of the two following principles.

C. *In a spherical right triangle a leg and its opposite angle are always in the same quadrant.*

For we have, formula (3), $\cos \alpha = \sin \beta \cos a$, and since $\sin \beta$ is always positive ($\beta < 180°$), $\cos \alpha$ and $\cos a$ must have the same sign; that is, α and a must be either both less or both greater than 90°.

D. *When the two legs are both in the same quadrant the hypotenuse is less than 90°; and when the legs are in different quadrants the hypotenuse is greater than 90°.*

This follows from formula (1), $\cos c = \cos a \cos b$. For if a and b are in the same quadrant, $\cos a$ and $\cos b$ have like signs and their product, $\cos c$, is positive, and c is in the first quadrant, i.e., $c < 90°$. If a, b are in different quadrants their cosines have unlike signs and $\cos c$ is negative; hence c is in the second quadrant, i.e., $c > 90°$. The relations (C) and (D) are known as the *Rules of Species.*

E. *When the two given parts of a spherical right triangle are a leg and its opposite angle, there are always two solutions.*

For in this case (Case VI) we have given a, α or b, β and therefore use one of formulas (3), (7), (9) or (4), (8), (10); in each the unknown part is found from its sine, and may therefore be in either the first or second quadrant.

This is the only case in which the *Rules of Species* do not enable us to determine the quadrant in which the computed part lies. It is called the *ambiguous case* of spherical right triangles.

31. *Case I: Hypotenuse and Angle Given.*—Suppose the given parts are $c=129°\ 14.6'$, $\alpha=43°\ 15.7'$. The required parts are then a, b, β, and the three solution formulas, each containing both the given parts and only one required part are the formulas (7), (5), (2), respectively:

$$\sin a=\sin c \sin \alpha,\ \cos \alpha=\tan b \cot c,\ \cos c=\cot \alpha \cot \beta.$$

Solving these for the required parts:

$$\sin a=\sin c \sin \alpha, \quad \tan b=\tan c \cos \alpha, \quad \cot \beta=\cos c \tan \alpha.$$

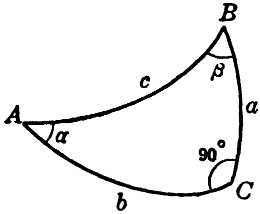
The logarithmic solutions are, therefore:

$$\log \sin a=\log \sin c+\log \sin \alpha,$$

$$\log \tan b=\log \tan c+\log \cos \alpha,$$

$$\log \cot \beta=\log \cos c+\log \tan \alpha.$$

The work is laid out as follows, the figure showing the relation of the parts. The figure should always be drawn, but the logarithmic formulas may be omitted, as the solution formulas show what operations are to be performed. This form should always be used.



$c = 129° \, 14' \, 24''$
$\alpha = 43° \, 15' \, 42''$

$a = 32° \, 3' \, 24''$
$b = 138° \, 16' \, 58''$
$\beta = 120° \, 46' \, 2''$

$\sin a = \sin c \sin \alpha$
$\tan b = \tan c \cos \alpha$
$\cot \beta = \cos c \tan \alpha$

$\log \sin 129° \, 14' \, 24'' = 9.88900 - 10$
$(+) \log \sin 43° \, 15' \, 42'' = 9.83590 - 10$

$\log \sin a = 9.72490 - 10$
$a = 32° \, 3' \, 24''$

$\log \tan 129° \, 14' \, 24'' = 10.08786 - 10(n)$
$(+) \log \cos 43° \, 15' \, 42'' = 9.86227 - 10$

$\log \tan b = 9.95013 - 10(n)$
$b = 138° \, 16' \, 58''$

$\log \cos 129° \, 14' \, 24'' = 9.80114 - 10(n)$
$(+) \log \tan 43° \, 15' \, 42'' = 9.97363 - 10$

$\log \cot \beta = 9.77477 - 10(n)$
$\beta = 120° \, 46' \, 2''$

The complete form should be laid out before any logarithms are read from the tables, the spaces for the required parts being provided, to be filled in after the computations are completed.

The logarithms should all be read and written down before the computations are begun.

All the computations should then be performed before any anti-logarithms are read.

Each anti-logarithm should be written in its proper place in the right hand column as it is read, and finally all three results should be copied in the spaces provided in the left column.

In the model solution given above the sign in parentheses preceding a logarithm indicates whether that logarithm is to be added or subtracted.

The letter (n) following a logarithm indicates that the *natural* function is *negative*. All functions are to be treated as if they were positive in the computations and the sign of the result determined by the usual algebraic rules of signs in the solution formulas at the left, those which are negative being indicated by the symbol (n).

Thus in the first solution formula in the solution above, both factors are positive and hence the product is positive; in each of the second and third, one factor is negative and hence the product is negative.

The sign of the product functions tangent and cotangent show that b and β are in the second quadrant ($>90°$).

In determining the quadrant for a we must use (C), Art. 30. Here α is in the first quadrant ($43° \; 15' \; 42''$) and hence also $a<90°$.

The check formula for the solution just finished, containing the three computed parts, is formula (10), Art. 28. Every solution should be checked by the appropriate formula, using the logs already used. Thus here it is found that $\log \sin a = \log \tan b + \log \cot \beta$.

Exercises

Solve the following spherical right triangles by the method illustrated above.

1. $c=110°$, $\beta=48° \; 28' \; 36''$.
2. $c=75° \; 20' \; 30''$, $\alpha=55° \; 18' \; 13''$.
3. $c=54° \; 20'$, $\alpha=46° \; 49' \; 43''$.
4. $c=87° \; 11' \; 40''$, $\beta=32° \; 42' \; 39''$.
5. $c=69° \; 25' \; 11''$, $\alpha=54° \; 54' \; 42''$.
6. $c=112° \; 48'$, $\alpha=56° \; 11' \; 56''$.
7. $c=46° \; 40' \; 12''$, $\alpha=37° \; 46' \; 9''$.
8. $c=118° \; 40' \; 1''$, $\alpha=128° \; 0' \; 4''$.
9. $c=110° \; 46' \; 20''$, $\alpha=80° \; 10' \; 30''$.
10. $c=98° \; 14' \; 24''$, $\beta=101° \; 47' \; 56''$.
11. $c=87° \; 58'$, $\alpha=34° \; 7' \; 41''$.
12. $c=37° \; 40' \; 20''$, $\beta=0° \; 43' \; 32''$.

32. *Case II: Hypotenuse and Leg Given.*—As the solution formulas (1)–(10) of Art. 28 are all of the same form, the solutions of all cases of spherical right triangles are very similar in form, and detailed solutions will, therefore, not be given here for every case. The solution formulas will be given for every case for one choice of the given parts. If a different pair of the parts is given in an exercise the formulas may be easily selected from the list in Art. 28 or written out by means of Napier's Rules.

In Case II the given parts are the hypotenuse and a leg. Sup-

pose we have given c and a. The required parts are b, α, β and the proper formulas of Art. 28 are (1), (7), (6):

$$\cos c=\cos a \cos b, \quad \sin a=\sin c \sin \alpha, \quad \cos \beta=\tan a \cot c.$$

Solving for the required parts:

$$\cos b=\frac{\cos c}{\cos a}, \quad \sin \alpha=\frac{\sin a}{\sin c}, \quad \cos \beta=\frac{\tan a}{\tan c}$$

are the solution formulas. The logarithmic forms are

$$\log \cos b=\log \cos c-\log \cos a,$$

$$\log \sin \alpha=\log \sin a-\log \sin c,$$

$$\log \cos \beta=\log \tan a-\log \tan c.$$

The work is to be laid out and the solution carried through as in Art. 31, using 5-place tables and reading the results to the nearest whole second of angle or arc.

The check formula is (4), $\cos \beta=\sin \alpha \cos b$, or $\log \sin \alpha+\log \cos b=\log \cos \beta$, and the sum of the first two logarithms, as used or found in the solution, must be equal to the third in this equation.

33. *Case III: Two Angles Given.*—We have given α, β, to find a, b, c. The formulas are $\cos \alpha=\sin \beta \cos a$, $\cos \beta=\sin \alpha \cos b$, $\cos c=\cot \alpha \cot \beta$, or, solved,

$$\cos a=\frac{\cos \alpha}{\sin \beta}, \quad \cos b=\frac{\cos \beta}{\sin \alpha}, \quad \cos c=\cot \alpha \cot \beta.$$

The logarithmic formulas are written out from these at once.

34. *Case IV: Legs Given.*—Given a, b; to find α, β, c. The formulas are

$$\sin b=\tan a \cot \alpha, \quad \sin a=\tan b \cot \beta, \quad \cos c=\cos a \cos b.$$

$$\therefore \cot \alpha=\frac{\sin b}{\tan a}, \quad \cot \beta=\frac{\sin a}{\tan b}, \quad \cos c=\cos a \cos b,$$

with the corresponding logarithmic formulas.

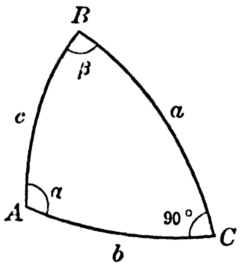
35. *Case V: Angle and Adjacent Leg Given.*—Suppose a, β given, to find α, b, c. The final solution formulas are

$$\cos \alpha = \sin \beta \cos a, \quad \tan b = \sin a \tan \beta, \quad \cot c = \frac{\cos \beta}{\tan a}.$$

36. *Case VI: Angle and Opposite Leg Given.*—This is the ambiguous case and an example will be worked out. Suppose $a = 160° \ 12.2'$, $\alpha = 150° \ 37'$, to find b, c, β. The final solution formulas are

$$\sin b = \frac{\tan a}{\tan \alpha}, \quad \sin c = \frac{\sin a}{\sin \alpha}, \quad \sin \beta = \frac{\cos \alpha}{\cos a}.$$

One of the figures is given.



$a = 160° \ 12' \ 12''$
$\alpha = 150° \ 37'$

$b = 39° \ 44' \ 6''$
$c = 136° \ 20' \ 54''$
$\beta = 67° \ 52' \ 12''$

$b' = 140° \ 15' \ 54''$
$c' = 43° \ 39' \ 6''$
$\beta' = 112° \ 9' \ 48''$

$$\sin b = \frac{\tan a}{\tan \alpha}$$

$$\sin c = \frac{\sin a}{\sin \alpha}$$

$$\sin \beta = \frac{\cos \alpha}{\cos a}$$

$\log \tan 160° \ 12' \ 12'' = 9.55625 - 10(n)$
$(-)\log \tan 150° \ 37' = 9.75058 - 10(n)$

$\log \sin b = 9.80567 - 10$
$b = 39° \ 44' \ 6''$
$b' = 140° \ 15' \ 54''$

$\log \sin 160° \ 12' \ 12'' = 9.52979 - 10$
$(-)\log \sin 150° \ 37' = 9.69077 - 10$

$\log \sin c = 9.83902 - 10$
$c' = 43° \ 39' \ 6''$
$c = 136° \ 20' \ 54''$

$\log \cos 150° \ 37' = 9.94020 - 10(n)$
$(-)\log \cos 160° \ 12' \ 12'' \mp 9.97354 - 10(n)$

$\log \sin \beta = 9.96666 - 10$
$\beta = 67° \ 50' \ 12''$
$\beta' = 112° \ 9' \ 48''$

In the first formula both tangents are negative; hence sin b is positive. In the second all three sines are positive. In the third both cosines are negative, and sin β is positive.

Both values of each computed result must be used (Art. 30). The three which are to be used together to form each solution are determined by the *Rules of Species*, as follows:

One solution: $\alpha > 90°, a > 90°;\quad \beta < 90°, b < 90°$ (C);

hence $c > 90°$, by (D).

Other solution: $\alpha > 90°, a > 90°;\quad \beta > 90°, b > 90°$ (C);

hence $c < 90°$, by (D).

Thus if the values marked b, c, β are chosen as one solution, their supplements b', c', β' must form the other solution.

37. *Solution of Quadrantal Triangles.*—A quadrantal triangle has a side (or sides) equal to a quadrant. The *polar* of a quadrantal triangle (Art. 15) therefore has a right angle, the supplement of the quadrant side, and may therefore be solved by the methods already given. The required parts of the given quadrantal triangle are then found as the supplements of the corresponding computed parts of the polar.

The exercises which follow include all cases of spherical right triangles, including also quadrantal triangles. They are to be solved by the methods of this chapter.

Exercises

1. $a=36^\circ\ 27'$
   $b=43^\circ\ 32'\ 31''$.
2. $a=86^\circ\ 40'$
   $b=32^\circ\ 40'$.
3. $a=50^\circ$
   $b=36^\circ\ 54'\ 49''$.
4. $a=120^\circ\ 10'$
   $b=150^\circ\ 59'\ 44''$.
5. $c=55^\circ\ 9'\ 32''$
   $a=22^\circ\ 15'\ 7''$.
6. $c=23^\circ\ 49'\ 51''$
   $a=14^\circ\ 16'\ 35''$.
7. $c=44^\circ\ 33'\ 17''$
   $a=32^\circ\ 9'\ 17''$.
8. $c=97^\circ\ 13'\ 4''$
   $a=132^\circ\ 14'\ 12''$.
9. $a=77^\circ\ 21'\ 50''$
   $\alpha=83^\circ\ 56'\ 40''$.
10. $a=77^\circ\ 21'\ 50''$
    $\alpha=40^\circ\ 40'\ 40''$.
11. $a=92^\circ\ 47'\ 32''$
    $\beta=50^\circ\ 2'\ 1''$.
12. $a=2^\circ\ 0'\ 55''$
    $\beta=12^\circ\ 40'$.
13. $a=20^\circ\ 20'\ 20''$
    $\beta=38^\circ\ 10'\ 10''$.
14. $a=54^\circ\ 30'$
    $\beta=35^\circ\ 30'$.
15. $\alpha=63^\circ\ 15'\ 12''$
    $\beta=135^\circ\ 33'\ 39''$.
16. $\alpha=116^\circ\ 43'\ 12''$
    $\beta=116^\circ\ 31'\ 25''$.
17. $\alpha=46^\circ\ 59'\ 42''$
    $\beta=57^\circ\ 59'\ 17''$.
18. $\alpha=90^\circ$
    $\beta=88^\circ\ 24'\ 35''$.
19. $c=75^\circ\ 0'\ 24''$
    $a=32^\circ\ 56'$.
20. $c=100^\circ\ 12'$
    $b=40^\circ\ 30'\ 12''$.
21. $\alpha=30^\circ\ 51\ 2'$
    $\beta=71^\circ\ 36'$.
22. $\alpha=130^\circ\ 20'$
    $\beta=100^\circ\ 10.9'$.
23. $a=43^\circ\ 20'$
    $b=74^\circ\ 13'$.
24. $a=100^\circ$
    $b=98^\circ\ 20'$.
25. $b=66^\circ\ 29'$
    $\alpha=50^\circ\ 17'$.
26. $a=24^\circ\ 41'$
    $\beta=140^\circ\ 34.7'$.
27. $a=40^\circ\ 42.4'$
    $c=63^\circ\ 20'$.
28. $a=70^\circ\ 15.5'$
    $\alpha=81^\circ\ 42\ 7'$.
29. $b=30^\circ\ 32.4'$
    $\alpha=36^\circ\ 44'$.
30. $c=72^\circ\ 10'$
    $\alpha=30^\circ\ 43'$.
31. $\alpha=106^\circ\ 34.2'$
    $\beta=33^\circ\ 11.7'$.
32. $a=28^\circ\ 47'$
    $b=110^\circ\ 27.3'$.
33. $c=54^\circ\ 12.2'$
    $\beta=164^\circ\ 50.4'$.
34. $a=40^\circ\ 8'$
    $\beta=74^\circ\ 30.2'$.
35. $c=102^\circ\ 36'$
    $\alpha=125^\circ\ 13'\ 24''$.
36. $\alpha=40^\circ\ 42'\ 24''$
    $\beta=67^\circ\ 51'\ 36''$.
37. $b=163^\circ\ 14'\ 12''$
    $c=112^\circ\ 41'\ 48''$.
38. $a=130^\circ\ 30'\ 12''$
    $b=140^\circ\ 12'$.
39. $c=50^\circ\ 20'\ 12''$
    $\beta=101^\circ\ 29'\ 24''$.
40. $b=10^\circ\ 10'\ 12''$
    $\beta=15^\circ\ 40'\ 24''$.
41. $c=90^\circ$
    $\gamma=98^\circ\ 22'\ 42''$
    $\alpha=150^\circ\ 47'$.
42. $c=90^\circ$
    $\alpha=121^\circ\ 30'$
    $\beta=112^\circ\ 16'\ 12''$.
43. $c=90^\circ$
    $a=138^\circ\ 47'\ 48''$
    $b=107^\circ\ 54'\ 54''$.
44. $c=90^\circ$
    $a=112^\circ\ 6'\ 30''$
    $\gamma=74^\circ\ 30'$.
45. $c=90^\circ$
    $a=94^\circ\ 22'\ 12''$
    $\alpha=108^\circ\ 13'\ 18''$.

CHAPTER V

SOLUTION OF OBLIQUE SPHERICAL TRIANGLES

38. *Solutions of Oblique Spherical Triangles.*—An *oblique* spherical triangle is a spherical triangle which has no right angle. Each of the six parts may therefore have any value less than 180°.

It is shown in geometry that any such triangle may be constructed (in trigonometry, *solved*) when any three of its six parts are given. There are six ways in which three of the six parts may be given:

Case I. The three sides.
II. The three angles.
III. Two sides and their included angle.
IV. Two angles and their included side.
V. Two sides and the angle opposite one.
VI. Two angles and the side opposite one.

In certain applications it may be necessary to find only one or two of the three unknown parts, and for such cases special methods and formulas are developed from the formulas of Chapter III. In this chapter, however, we shall give only the standard complete solution for each case, except Case III.

These will be taken up in order and an illustrative solution worked out for each, after the solution formulas are developed.

39. *Solution Formulas.*—Any oblique spherical triangle may be solved by means of the following formulas, which are derived in Chapter III.

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}. \tag{1}$$

$$\left.\begin{aligned}\tan\frac{\alpha}{2}&=\frac{r}{\sin(s-a)}, \quad s=\tfrac{1}{2}(a+b+c),\\ \tan r&=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},\end{aligned}\right\} \tag{2}$$

and two similar formulas for $\frac{\beta}{2}, \frac{\gamma}{2}$.

$$\left.\begin{aligned}\tan\frac{a}{2}&=\tan R\cos(S-\alpha), \quad S=\tfrac{1}{2}(\alpha+\beta+\gamma),\\ \tan R&=\sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}},\end{aligned}\right\} \tag{3}$$

and two similar formulas for $\frac{b}{2}, \frac{c}{2}$.

$$\tan\tfrac{1}{2}(a-b)=\frac{\tan\frac{c}{2}\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}, \tag{4}$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\tan\frac{c}{2}\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}, \tag{5}$$

$$\tan\tfrac{1}{2}(\alpha-\beta)=\frac{\cot\frac{\gamma}{2}\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}, \tag{6}$$

$$\tan\tfrac{1}{2}(\alpha+\beta)=\frac{\cot\frac{\gamma}{2}\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}, \tag{7}$$

and formulas similar to the last four for b, c and c, a; β, γ and γ, α.

The spherical excess, E, of any spherical triangle may be found

from the given or computed parts by means of the following formulas:

$$E = \alpha + \beta + \gamma - 180^\circ. \tag{8}$$

$$\cot\frac{E}{2} = \frac{\cot\frac{a}{2}\cot\frac{b}{2}}{\sin\gamma} + \cot\gamma, \tag{9}$$

and two similar formulas for b, c, α and c, a, β.

$$\tan\frac{E}{4} = \sqrt{\tan\frac{s}{2}\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}. \tag{10}$$

With E expressed in degrees the area of the triangle on a sphere of radius R or diameter D is

$$K = \left(\frac{\pi E}{180}\right)R^2 = \left(\frac{\pi E}{720}\right)D^2. \tag{11}$$

When E is expressed in radians

$$K = ER^2 = \tfrac{1}{4}ED^2. \tag{12}$$

The lengths of the sides of the triangle are

$$\bar{a} = Ra, \quad \bar{b} = Rb, \quad \bar{c} = Rc \tag{13}$$

when a, b, c are expressed in radians.

40. *Quadrant in which a Computed Part Lies.*—The quadrant in which a computed part of an oblique spherical triangle lies can always be determined by means of two rules:

A. *If a side* (*or angle*) *differs more than another side* (*or angle*) *from 90° it lies in the same quadrant as its opposite angle* (*or side*).

To prove this, we have from the first fundamental formula (3), Art. 17,

$$\cos\alpha = \frac{\cos a - \cos b\cos c}{\sin b\sin c}. \tag{14}$$

Since b and c are each $<180°$ sin b, sin c are both positive and hence the denominator of the fraction in (14) is positive. The sign of the fraction, and hence of cos α, is therefore the same as that of the numerator.

But if a differs from 90° more than b, cos $a>$cos b, numerically, and hence cos $a>$cos b cos c, since cos $c<1$. Therefore the numerator has the same sign as cos a.

Therefore cos a and cos α have the same sign, and hence a, α are in the same quadrant. Similarly b, β would be in the same quadrant, and also c, γ.

We have also, from the first of (8), Art. 20,

$$\cos a=\frac{\cos\alpha+\cos\beta\cos\gamma}{\sin\beta\sin\gamma},$$

and reasoning from this formula as above from (14), it is shown that when α differs from 90° more than β or γ, α and a are in the same quadrant; and similarly for β, b and γ, c.

B. *Half the sum of two sides must lie in the same quadrant as half the sum of the two opposite angles.*

For, from (5),

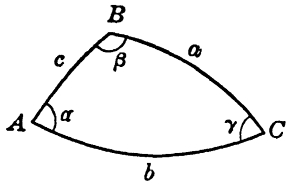
$$\tan\tfrac{1}{2}(a+b)\cdot\cos\tfrac{1}{2}(\alpha+\beta)=\tan\frac{c}{2}\cdot\cos\tfrac{1}{2}(\alpha-\beta), \tag{15}$$

and in this $\dfrac{c}{2}<90°$ and $\alpha-\beta<180°$, or $\frac{1}{2}(\alpha-\beta)<90°$. Hence both factors on the right of (15), and also their product, are positive. Therefore the product on the left of (15) is positive; hence tan $\frac{1}{2}(a+b)$ and cos $\frac{1}{2}(\alpha+\beta)$ have the same sign, and $\frac{1}{2}(a+b)$, $\frac{1}{2}(\alpha+\beta)$ are both in the first or both in the second quadrant.

Similarly the half sum of b, c and β, γ are in the same quadrant; and likewise c, a and γ, α.

(A) and (B) are called the *Rules of Species* for oblique spherical triangles.

41. *Case I: Three Sides Given.*—In this case the three angles are found by means of the formulas (2), Art. 39. The logarithmic operations are obvious from the formulas, and the work is laid out as follows, given $a=114°\ 43'\ 18''$, $b=136°\ 19'\ 36''$, $c=43°\ 18'\ 30''$.



$$a=114°\ 43'\ 18''$$
$$b=136°\ 19'\ 36''$$
$$c=43°\ 18'\ 30''$$

$$\alpha=76°\ 48'\ 24''$$
$$\beta=132°\ 15'\ 12''$$
$$\gamma=47°\ 19'\ 24''$$

$$s=\tfrac{1}{2}(a+b+c)$$

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan\frac{\alpha}{2}=\frac{r}{\sin(s-a)}$$

$$\tan\frac{\beta}{2}=\frac{r}{\sin(s-b)}$$

$$\tan\frac{\gamma}{2}=\frac{r}{\sin(s-c)}$$

$$a=114°\ 43'\ 18''$$
$$b=136°\ 19'\ 36''$$
$$c=43°\ 18'\ 30''$$

$$a+b+c=294°\ 21'\ 24''$$
$$s=147°\ 10'\ 42''$$
$$s-a=32°\ 27'\ 24''$$
$$s-b=10°\ 51'\ 6''$$
$$s-c=103°\ 52'\ 12''$$

$$\log\sin 32°\ 27'\ 24''=9.72970-10$$
$$(+)\log\sin 10°\ 51'\ 6''=9.27478-10$$
$$(+)\log\sin 103°\ 52'\ 12''=9.98714-10$$

$$\log \text{Prod}=28.99162-30$$
$$(-)\log\sin 147°\ 10'\ 42''=9.73402-10$$

$$\log\tan^2 r=2\log\tan r=19.25760-20$$
$$\log\tan r=9.62880-10$$
$$(-)\log\sin 32°\ 27'\ 24''=9.72970-10$$
$$(-)\log\sin 10°\ 51'\ 6''=9.27478-10$$
$$(-)\log\sin 103°\ 52'\ 12''=9.98714-10$$

$$\log\tan\frac{\alpha}{2}=9.89910-10$$

$$\log\tan\frac{\beta}{2}=0.35402$$

$$\log\tan\frac{\gamma}{2}=9.64166-10$$

$$\frac{\alpha}{2}=38°\ 24'\ 12''$$

$$\frac{\beta}{2}=66°\ 7'\ 36''$$

$$\frac{\gamma}{2}=23°\ 39'\ 42''$$

The procedure in making out the form (including finding s, $s-a$, $s-b$, $s-c$) and reading the logarithms and anti-logarithms, is the same as that described in Art. 31. The values of $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ are the last values read from the table, and these are entered on the solution sheet in the position shown, as they are read. Finally each of these three values is doubled and the results entered in the left hand column in the spaces already provided for α, β, γ.

The check formula is the sine proportion (1), used in the logarithmic form, $(\log \sin a - \log \sin \alpha) = (\log \sin b - \log \sin \beta) = (\log \sin c - \log \sin \gamma)$. If these three differences are equal the solution is correct.

The spherical excess of the triangle is $E = (\alpha+\beta+\gamma) - 180° = 256° \ 23' - 180° = 76° \ 23'$. It is computed independently from the given data (a, b, c) by formula (10),

$$\tan \frac{E}{4} = \sqrt{\tan \frac{s}{2} \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)},$$

as follows, using the values of s, $s-a$, $s-b$, $s-c$ found in the solution of the triangle.

$$\frac{s}{2} = 73° \ 35' \ 21''$$
$$\tfrac{1}{2}(s-a) = 16° \ 13' \ 42''$$
$$\tfrac{1}{2}(s-b) = 5° \ 25' \ 33''$$
$$\tfrac{1}{2}(s-c) = 51° \ 56' \ 6''$$

$$\begin{aligned} \log \tan 73° \ 35' \ 21'' &= 0.53089 \\ (+)\log \tan 16° \ 13' \ 42'' &= 9.46399-10 \\ (+)\log \tan 5° \ 25' \ 33'' &= 8.97764-10 \\ (+)\log \tan 51° \ 56' \ 6'' &= 0.10618 \end{aligned}$$

$$\log \tan^2 \frac{E}{4} = 2 \log \tan \frac{E}{4} = 19.07870-20$$

$$\log \tan \frac{F}{4} = 9.53935-10$$

$$\frac{E}{4} = 19° \ 5' \ 45''$$

$$E = 76° \ 23' \ 0''$$

This is the value found as $\alpha+\beta+\gamma-180°$.

If E is to be used for computing the area of the triangle, it must be expressed in degrees alone. Thus $76° 23' = 76\frac{23}{60}° = 76.383°$.

When the radius R of the sphere is given, the area of this triangle is

$$K=\left(\frac{76.383\pi}{180}\right)R^2=1.3332R^2,$$

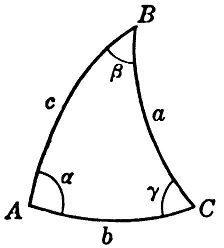
in square units of the kind in which R is expressed.

Exercises

Solve and check the following triangles, compute the spherical excess from the sides and check by the angles, and find the area on a sphere of two feet diameter.

No	*a*	*b*	*c*	No	*a*	*b*	*c*
1	120° 55′ 35″	59° 4′ 25″	106° 10′ 25″	10	100°	50°	60°
2	50° 12′ 4″	116° 44′ 48″	129° 11′ 42″	11	10°	7°	4°
3	131° 35′ 4″	108° 30′ 14″	84° 46′ 34″	12	70° 14′ 20″	49° 24′ 10″	38° 46′ 10″
4	20° 16′ 38″	56° 19′ 40″	66° 20′ 44″	13	138° 4′	109° 41′	90°
5	124° 12′ 31″	54° 18′ 16″	97° 12′ 25″	14	76° 35′ 36″	50° 10′ 30″	40° 0′ 10″
6	76° 40 4′	54° 21 3′	36° 8 7′	15	56° 37′	108° 14′	75° 29′
7	124° 34 9′	66° 7 2′	109° 43 5″	16	143° 46′	67° 24′	132° 11′
8	30° 17 6′	22° 14 4′	18° 51 8′	17	33° 4′	74° 16′	94° 18′
9	130° 46 0′	113° 21 4′	102° 16 2′	18	62° 54′ 4″	125° 20′	131° 30′

42. *Case II: Three Angles Given.*—In this case the three sides are found by formulas (3). Suppose $\alpha=116°\ 19'\ 24''$, $\beta=83°\ 19'\ 12''$, $\gamma=106°\ 10'\ 36''$. The solution follows, and is checked by the sine law.



$$\alpha=116°\ 19'\ 24''$$
$$\beta=\ \ 83°\ 19'\ 12''$$
$$\gamma=106°\ 10'\ 36''$$

$$a=119°\ 55'\ 24''$$
$$b=\ \ 73°\ 49'\ 12''$$
$$c=111°\ 46'\ 12''$$

$$S=\tfrac{1}{2}(\alpha+\beta+\gamma)$$

$$\tan R=\sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}$$

$$\tan\frac{a}{2}=\tan R\cos(S-\alpha)$$

$$\tan\frac{b}{2}=\tan R\cos(S-\beta)$$

$$\tan\frac{c}{2}=\tan R\cos(S-\gamma)$$

$$\alpha=116°\ 19'\ 24''$$
$$\beta=\ \ 83°\ 19'\ 12''$$
$$\gamma=106°\ 10'\ 36''$$

$$\alpha+\beta+\gamma=305°\ 49'\ 12''$$
$$S=152°\ 54'\ 36''$$
$$S-\alpha=\ \ 36°\ 35'\ 12''$$
$$S-\beta=\ \ 69°\ 35'\ 24''$$
$$S-\gamma=\ \ 46°\ 44'\ \ 0''$$

$$\log(-\cos 152°\ 54'\ 36'')=9.94953-10$$
$$\log\cos\ 36°\ 35'\ 12''=9.90469-10$$
$$(+)\log\cos\ 69°\ 35'\ 24''=9.54249-10$$
$$(+)\log\cos\ 46°\ 44'\ \ 0''=9.83594-10$$

$$(-)\log \text{Denom}=9.28312-10$$
$$\log\tan^2 R=2\log\tan R=0.66641$$
$$\log\tan R=0.33320$$
$$(+)\log\cos 36°\ 35'\ 12''=9.90469-10$$
$$(+)\log\cos 69°\ 35'\ 24''=9.54249-10$$
$$(+)\log\cos 46°\ 44'\ \ 0''=9.83594-10$$

$$\log\tan\frac{a}{2}=0.23789$$

$$\log\tan\frac{b}{2}=9.87569-10$$

$$\log\tan\frac{c}{2}=0.16914$$

$$\frac{a}{2}=59°\ 57'\ 42''$$

$$\frac{b}{2}=36°\ 54'\ 36''$$

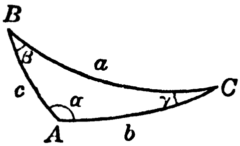
$$\frac{c}{2}=55°\ 53'\ \ 6''$$

Exercises

Solve and check the following triangles, and find the area and lengths of sides on a two-foot globe.

No	α	β	γ	No	α	β	γ
1	129° 5′ 28″	142° 12′ 42″	105° 8′ 10″	7	4° 23′ 35″	8° 28′ 20″	172° 17′ 56″
2	111° 4′	143° 18′	31° 30′	8	130°	110°	80°
3	70° 39′	48° 36′	119° 15′	9	110° 36 4′	122° 8 7′	140° 20 3′
4	114° 30′	83° 12′	123° 20′	10	120° 50 6′	78° 6 1′	81° 12 3′
5	59° 55′ 10″	85° 36′ 50″	59° 55′ 10″	11	80° 20 2′	73° 46 7′	54° 8 5′
6	102° 14′ 12″	54° 32′ 24″	89° 5′ 46″	12	100° 51 3′	80° 47 6′	74° 3 3′

43. *Case III: Two Sides and Included Angle Given. First Solution.*—Suppose b, c, α given, to find β, γ, a. The solution formulas are (6), (7) with β, γ and b, c replacing α, β and a, b; and (4) with parts permuted to give a. The first two give $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\beta-\gamma)$, and $\beta, \gamma=\frac{1}{2}(\beta+\gamma)\pm\frac{1}{2}(\beta-\gamma)$. The third gives $\tan\frac{a}{2}$ directly. Given $b=105°\ 14'\ 48''$, $c=43°\ 17'\ 12''$, $\alpha=112°\ 47'\ 24''$, the solution follows, and is checked by the sine law.



$b = 105°\ 14'\ 48''$
$c = 43°\ 17'\ 12''$
$\alpha = 112°\ 47'\ 24''$

$\beta = 84°\ 6'\ 42''$
$\gamma = 44°\ 59'\ 6''$
$a = 116°\ 35'\ 36''$

$$\tan\tfrac{1}{2}(\beta+\gamma) = \frac{\cot\frac{\alpha}{2}\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}$$

$$\tan\tfrac{1}{2}(\beta-\gamma) = \frac{\cot\frac{\alpha}{2}\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}$$

$$\beta, \gamma = \tfrac{1}{2}(\beta+\gamma) \pm \tfrac{1}{2}(\beta-\gamma)$$

$$\tan\frac{a}{2} = \frac{\tan\frac{1}{2}(b-c)\sin\frac{1}{2}(\beta+\gamma)}{\sin\frac{1}{2}(\beta-\gamma)}$$

$b = 105°\ 14'\ 48''$	$\alpha = 112°\ 47'\ 24''$
$c = 43°\ 17'\ 12''$	$\frac{\alpha}{2} = 56°\ 23'\ 42''$
$b+c = 148°\ 32'\ 0''$	$\frac{1}{2}(b+c) = 74°\ 16'\ 0''$
$b-c = 61°\ 57'\ 36''$	$\frac{1}{2}(b-c) = 30°\ 58'\ 48''$

$\log\cot 56°\ 23'\ 42'' = 9.82251-10$
$(+)\log\cos 30°\ 58'\ 48'' = 9.93316-10$

$\log \text{Prod.} = 9.75567-10$
$(-)\log\cos 74°\ 16'\ 0'' = 9.43323-10$

$\log\tan\frac{1}{2}(\beta+\gamma) = 0.32244$
$\rightarrow \frac{1}{2}(\beta+\gamma) = 64°\ 32'\ 54''$

$\log\cot 56°\ 23'\ 42'' = 9.82251-10$
$(+)\log\sin 30°\ 58'\ 48'' = 9.71159-10$

$\log \text{Prod} = 19.53410-20$
$(-)\log\sin 74°\ 16'\ 0'' = 9.98342-10$

$\log\tan\frac{1}{2}(\beta-\gamma) = 9.55068-10$
$\rightarrow (\pm)\frac{1}{2}(\beta-\gamma) = 19°\ 33'\ 48''$

$\beta = 84°\ 6'\ 42''$
$\gamma = 44°\ 59'\ 6''$

$\log\tan 30°\ 58'\ 48'' = 9.77843-10$
$(+)\log\sin 64°\ 32'\ 54'' = 9.95556-10$

$\log \text{Prod.} = 9.73399-10$
$(-)\log\sin 19°\ 33'\ 48'' = 9.52486-10$

$\log\tan\frac{a}{2} = 0.20913$

$\frac{a}{2} = 58°\ 17'\ 48''$

The spherical excess is computed from the given parts by using formula (9), with a, b, γ replaced by b, c, α:

$$\cot\frac{E}{2}=\frac{\cot\frac{b}{2}\cot\frac{c}{2}}{\sin\alpha}+\cot\alpha$$

$$\frac{b}{2}=52°\ 37'\ 24''$$

$$\frac{c}{2}=21°\ 38'\ 36''$$

$$\alpha=112°\ 47'\ 24''$$

$\log\cot\ 52°\ 37'\ 24''=$	$9.88305-10$
$(+)\log\cot\ 21°\ 38'\ 36''=$	0.40143
$\log\text{Prod.}=$	$10.28448-10$
$(-)\log\sin\ 112°\ 47'\ 24''=$	$9.96470-10$
$\log\text{Quot.}=$	0.31978
$\text{Quot.}=$	2.08824
$(+)\cot\ 112°\ 47'\ 24''=$	$0.42016(n)$
$\cot\frac{E}{2}=$	1.66808

$$\frac{E}{2}=30°\ 56'\ 35''$$

$$E=61°\ 53'\ 10''$$

The value obtained directly from the given and computed angles is

$$E=\alpha+\beta+\gamma-180°=241°\ 53'\ 12''-180°=61°\ 53'\ 12''.$$

44. *Case III: Second Solution.*—The problem of Case III is of importance in geodesy, navigation, and astronomy but for these purposes only side a and angle β are required. Special tables

have therefore been computed (called "azimuth tables") in which the angle β can be read directly from the given parts and only the side a must be computed. This computation does not require the complete formulas used in Art. 43 and is much simplified by the use of the *haversine* function, defined in (5), Art. 6.

The required computing formula is obtained from the first of the *fundamental formulas* (3), Art. 17,

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \tag{16}$$

By definition $\text{hav } a = \frac{1}{2}(1 - \cos a)$. Therefore

$$\cos a = 1 - 2 \text{ hav } a, \quad \cos \alpha = 1 - 2 \text{ hav } \alpha.$$

Substituting these expressions for $\cos a$ and $\cos \alpha$ in (16),

$$\begin{aligned} 1 - 2 \text{ hav } a &= \cos b \cos c + \sin b \sin c\ (1 - 2 \text{ hav } \alpha) \\ &= (\cos b \cos c + \sin b \sin c) - 2 \sin b \sin c \text{ hav } \alpha \\ &= \cos (b - c) - 2 \sin b \sin c \text{ hav } \alpha. \end{aligned}$$

Similarly $\cos (b-c) = 1 - 2 \text{ hav } (b-c)$, and the last formula becomes

$$1 - 2 \text{ hav } a = 1 - 2 \text{ hav } (b-c) - 2 \sin b \sin c \text{ hav } \alpha.$$

$$\therefore \text{ hav } a = \text{hav } (b-c) + \sin b \sin c \text{ hav } \alpha. \tag{17}$$

This is the required solution formula for a, given b, c, α. Tables of natural and logarithmic functions used by navigators and astronomers contain haversines and the solution for a is easily and rapidly carried out as follows.

In (17) let the last term

$$\left.\begin{aligned} \sin b \sin c \text{ hav } \alpha &= Q; \\ \text{then} \qquad \text{hav } a &= \text{hav } (b-c) + Q. \end{aligned}\right\} \tag{18}$$

In the example of Art. 43 we have

$$
\begin{array}{r}
\alpha = 112^\circ\ 47'\ 24'' \\
b = 105^\circ\ 14'\ 48'' \\
c = \ \ 43^\circ\ 17'\ 12'' \\
\hline
b-c = \ \ 61^\circ\ 57'\ 36''
\end{array}
$$

$$
\begin{array}{rr}
\log \sin\ \ 105^\circ\ 14'\ 48'' = & 9.98443-10 \\
(+)\log \sin\ \ \ \ 43^\circ\ 17'\ 12'' = & 9.83611-10 \\
(+)\log \text{hav}\ 112^\circ\ 47'\ 24'' = & 9.84116-10 \\
\hline
\log \text{Prod}\ Q = & 9.66170-10
\end{array}
$$

$$
\begin{array}{rr}
Q = & 0.45888 \\
(+)\text{hav}\ 61^\circ\ 57'\ 36'' = & 0.26495 \\
\hline
\text{hav}\ a = & 0.72383 \\
a = & 116^\circ\ 35'\ 36''
\end{array}
$$

This value of side a is the same as that found in Art. 43.

45. *Case IV: Two Angles and Included Side Given.*—Suppose α, β, c given, to find a, b, γ. The solution formulas are (4), (5), and (6) solved for $\cot \frac{\gamma}{2}$:

$$\tan \tfrac{1}{2}(a+b) = \frac{\tan \frac{c}{2} \cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)} \qquad (i)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\tan \frac{c}{2} \sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)}. \qquad (ii)$$

These two give $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$; then

$$a,\ b = \tfrac{1}{2}(a+b) \pm \tfrac{1}{2}(a-b). \qquad (iii)$$

Finally,

$$\cot \frac{\gamma}{2} = \frac{\tan \frac{1}{2}(\alpha-\beta) \sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)}. \qquad (iv)$$

These formulas are of the same form as those used in Case III, Art. 43, and the work is laid out and the logarithmic computation performed in the same manner, except that sides are used here where angles are used there, and vice versa. It is, therefore, not necessary to work out an illustrative solution here.

The solution is to be checked by the sine law in this case also.

Exercises

Solve and check the following triangles; compute the spherical excess from the given parts in Ex. 1-12, and find the area and lengths of sides in terms of the radius (R) in Ex. 13–24.

1. $b=120° \, 30' \, 30''$, $c=70° \, 20' \, 20''$, $\alpha=50° \, 10' \, 10''$.
2. $b=99° \, 40' \, 48''$, $c=100° \, 49' \, 30''$, $\alpha=65° \, 33' \, 10''$.
3. $a=68° \, 20' \, 25''$, $b=52° \, 18' \, 15''$, $\gamma=117° \, 12' \, 20''$.
4. $a=84° \, 14' \, 29''$, $b=44° \, 13' \, 45''$, $\gamma=36° \, 45' \, 28''$.
5. $a=89° \, 17'$, $c=52° \, 39'$, $\beta=119° \, 15'$.
6. $a=109° \, 21'$, $b=60° \, 45'$, $\gamma=127° \, 20' \, 56''$.
7. $a=73° \, 58' \, 54''$, $b=38° \, 45' \, 0''$, $\gamma=46° \, 33' \, 41''$.
8. $a=88° \, 12' \, 20''$, $b=124° \, 7' \, 17''$, $\gamma=50° \, 2' \, 1''$.
9. $a=120° \, 55' \, 35''$, $b=88° \, 12' \, 20''$, $\gamma=47° \, 42' \, 1''$.
10. $b=63° \, 15' \, 12''$, $c=47° \, 42' \, 1''$, $\alpha=59° \, 4' \, 25''$.
11. $b=69° \, 25' \, 11''$, $c=109° \, 46' \, 19''$, $\alpha=54° \, 54' \, 42''$.
12. $a=103° \, 44.7'$, $b=64° \, 12.3'$, $\gamma=98° \, 33.8'$.
13. $\alpha=135° \, 5' \, 29''$, $\gamma=50° \, 30' \, 9''$, $b=69° \, 34' \, 56''$.
14. $\alpha=95° \, 38' \, 4''$, $\gamma=97° \, 26' \, 29''$, $b=64° \, 23' \, 15''$.
15. $\alpha=130° \, 5' \, 22°$, $\beta=32° \, 26' \, 6''$, $c=51° \, 6' \, 12''$.
16. $\alpha=82° \, 27'$, $\beta=57° \, 30'$, $c=126° \, 37'$.
17. $\alpha=66° \, 57' \, 4''$, $\beta=97° \, 20' \, 32''$, $c=41° \, 9' \, 46''$.
18. $\alpha=107° \, 47' \, 7''$, $\beta=38° \, 58' \, 27''$, $c=51° \, 41' \, 14''$.
19. $\alpha=26° \, 58' \, 46''$, $\beta=39° \, 45' \, 10''$, $c=154° \, 46' \, 48''$.
20. $\alpha=128° \, 41' \, 49''$, $\beta=107° \, 33' \, 20''$, $c=124° \, 12' \, 31''$.
21. $\beta=153° \, 17' \, 6''$, $\gamma=78° \, 43' \, 36''$, $a=86° \, 15' \, 15''$.
22. $\alpha=125° \, 41' \, 44''$, $\gamma=82° \, 47' \, 35''$, $b=52° \, 37' \, 57''$.
23. $\beta=140° \, 43.2'$, $\gamma=100° \, 4.6'$, $a=60° \, 43.6'$.
24. $\alpha=140° \, 24.6'$, $\beta=12° \, 18.6'$, $c=28° \, 7.7'$.

46. *Case V: Two Sides and One Opposite Angle Given.*—Suppose a, b, α given, to find β, γ, c. The angle β is found at once by the appropriate proportion of the sine law (1), Art. 39,

$$\sin\beta = \frac{\sin\alpha \sin b}{\sin a}.$$

Then
$$\cot\frac{\gamma}{2} = \frac{\sin\frac{1}{2}(a+b)\tan\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(a-b)}$$

and
$$\tan\frac{c}{2} = \frac{\sin\frac{1}{2}(\alpha+\beta)\tan\frac{1}{2}(a-b)}{\sin\frac{1}{2}(\alpha-\beta)},$$

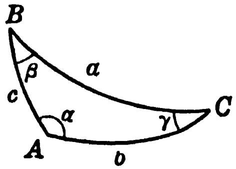
by (6) and (4), Napier's Analogies. The logarithmic computations are obvious, and the solution is checked by the complete sine law.

Since β is found by means of its sine it may be in either the first or second quadrant; hence there may be two solutions. If *b differs from 90° more than a*, β must be in the same quadrant as b, by Rule (A), and is therefore fixed, giving only one solution. However, if b does not differ from 90° more than a, Rule (A) does not determine β, and both values of β may be admissible. In this case Rule (B) will show whether this is so or not.

In any case there is no solution if the first computation gives $\sin\beta > 1$, or $\log\sin\beta > 0$ (positive).

The following solutions illustrate the preceding analysis: given $a = 148°\ 34'\ 24''$, $b = 142°\ 11'\ 36''$, $\alpha = 153°\ 17'\ 36''$.

Since $b - 90° < a - 90°$ there may be two solutions. The work is laid out as follows:



$$a = 148^\circ\, 34'\, 24''$$
$$b = 142^\circ\, 11'\, 36''$$
$$\alpha = 153^\circ\, 17'\, 36''$$

$$\beta = 31^\circ\, 53'\, 42''$$
$$\gamma = 6^\circ\, 17'\, 35''$$
$$c = 7^\circ\, 18'\, 20''$$

$$\beta' = 148^\circ\, 6'\, 18''$$
$$\gamma' = 130^\circ\, 21'\, 30''$$
$$c' = 62^\circ\, 8'\, 51''$$

$$\sin\beta = \frac{\sin\alpha \sin b}{\sin a}$$

$$\cot\frac{\gamma}{2} = \frac{\sin\frac{1}{2}(a+b)\tan\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(a-b)}$$

$$\tan\frac{c}{2} = \frac{\sin\frac{1}{2}(\alpha+\beta)\tan\frac{1}{2}(a-b)}{\sin\frac{1}{2}(\alpha-\beta)}$$

$$\begin{aligned}
\log\sin 153^\circ\, 17'\, 36'' &= 9.65265-10\\
(+)\log\sin 142^\circ\, 11'\, 36'' &= 9.78746-10\\
\hline
\log \text{Prod.} &= 19.44011-20\\
(-)\log\sin 148^\circ\, 34'\, 24'' &= 9.71718-10\\
\hline
\log\sin\beta &= 9.72293-10\\
\beta &= 31^\circ\, 53'\, 42''\\
\beta' &= 148^\circ\, 6'\, 18''
\end{aligned}$$

$a = 148^\circ\, 34'\, 24''$	$\alpha = 153^\circ\, 17'\, 36''$
$b = 142^\circ\, 11'\, 36''$	$\beta = 31^\circ\, 53'\, 42''$
$a+b = 290^\circ\, 46'\, 0''$	$\alpha+\beta = 185^\circ\, 11'\, 18''$
$a-b = 6^\circ\, 22'\, 48''$	$\alpha-\beta = 121^\circ\, 23'\, 54''$
$\frac{1}{2}(a+b) = 145^\circ\, 23'\, 0''$	$\frac{1}{2}(\alpha+\beta) = 92^\circ\, 35'\, 39''$
$\frac{1}{2}(a-b) = 3^\circ\, 11'\, 24''$	$\frac{1}{2}(\alpha-\beta) = 60^\circ\, 41'\, 57''$

$$\begin{aligned}
\log\sin 145^\circ\, 23'\, 0'' &= 9.75441-10\\
(+)\log\tan 60^\circ\, 41'\, 57'' &= 0.25089\\
\hline
\log \text{Prod.} &= 10.00530-10\\
(-)\log\sin 3^\circ\, 11'\, 24'' &= 8.74544-10\\
\hline
\log\cot\frac{\gamma}{2} &= 1.25986\\
\frac{\gamma}{2} &= 3^\circ\, 8'\, 47.4''\\
\gamma &= 6^\circ\, 17'\, 35''
\end{aligned}$$

$$\begin{aligned}
\log\sin 92^\circ\, 35'\, 39'' &= 9.99955-10\\
(+)\log\tan 3^\circ\, 11'\, 24'' &= 8.74612-10\\
\hline
\log \text{Prod.} &= 18.74567-20\\
(-)\log\sin 60^\circ\, 41'\, 57'' &= 9.94055-10\\
\hline
\log\tan\frac{c}{2} &= 8.80512-10\\
\frac{c}{2} &= 3^\circ\, 39'\, 10''\\
c &= 7^\circ\, 18'\, 20''
\end{aligned}$$

After finding the two values β, β' ($\beta'=180°-\beta$) the values $\frac{1}{2}(\alpha\pm\beta)$, $\frac{1}{2}(\alpha\pm\beta')$, $\frac{1}{2}(a\pm b)$ are computed and comparisons made according to Rule (B). It is found that both β and β' satisfy this rule, and hence both β and β' are solutions. Using $\beta=31°\ 53'\ 42''$ the computations are carried out above for γ, c. By using $\beta'=148°\ 6'\ 18''$ the values γ', c' may be found in the same manner.

Consider next the example: given $a=57°\ 36'$, $b=31°\ 14'$, $\alpha=104°\ 25'\ 30''$; to find β, γ and c.

In this example $a+b<180°$ and $\frac{1}{2}(a+b)<90°$; hence, according to Rule (B), $\frac{1}{2}(\alpha+\beta)<90°$, or $\alpha+\beta<180°$. But $\alpha>90°$, and hence $\beta<90°$. There is, therefore, only one solution. On carrying out the computation as in the preceding example, we find that $\log\sin\beta=9.77435-10$, and hence $\beta=36°\ 29'\ 46''<90°$ is the solution. When the computation is completed with this value of β it is found that $\gamma=51°\ 37'\ 56''$, $c=43°\ 7'\ 14''$.

47. *Case VI: Two Angles and One Opposite Side Given.*—Let the given parts be α, β, a, to find b, c, γ. The solution formulas are

$$\sin b=\frac{\sin a\sin\beta}{\sin\alpha},$$

$$\tan\frac{c}{2}=\frac{\tan\frac{1}{2}(a-b)\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)},$$

$$\cot\frac{\gamma}{2}=\frac{\sin\frac{1}{2}(a+b)\tan\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(a-b)}.$$

The numerical work is of precisely the same form as that of Art. 46 and an illustrative solution is unnecessary here.

This is also an ambiguous case, and the analysis is very similar to that of Case V. Thus, since b is found by means of its sine it it may be in either the first or second quadrants; hence there may be two solutions.

If β *differs from* 90° *more than* α, b must be in the same quadrant as β, and hence there is only one solution, Rule (A).

If, however, β does not differ from 90° more than α, Rule (A)

does not determine the quadrant for b and both values b, $b'=180°-b$ may be admissible. After finding both values Rule (B) will determine whether both are admissible.

As before, if $\sin b>1$, or $\log \sin b>0$ (positive), there is, of course, no solution.

The solutions of this case are checked by using the law of sines.

The following exercises include examples of both Cases V and VI. The appropriate formulas for combinations other than those used above (a, b, α or α, β, a) are immediately obvious. In any case the larger side and angle are to be written first in the sums and differences.

Exercises

Solve the following triangles where possible, determining in each case whether there are no, one, or two solutions.

1. $a=57° 36'$
   $b=31° 14'$
   $\alpha=104° 25' 30''$.
2. $a=73° 49' 38''$
   $b=120° 53' 35''$
   $\alpha=88° 52' 42''$.
3. $a=150° 57' 5''$
   $b=134° 15' 54''$
   $\alpha=144° 22' 42''$.
4. $a=79° 0' 54''$
   $b=82° 17' 4''$
   $\alpha=82° 9' 26''$.
5. $a=30° 52' 37''$
   $b=31° 9' 16''$
   $\alpha=87° 34' 12''$.
6. $b=40° 20.4'$
   $c=20° 18.2'$
   $\beta=60° 44.4'$.
7. $b=98° 16'$
   $c=74° 38'$
   $\beta=78° 40'$.
8. $a=64° 23' 15''$
   $c=99° 40' 48''$
   $\gamma=95° 38' 4''$.
9. $b=40° 5' 26''$
   $c=118° 22' 7''$
   $\beta=29° 42' 34''$.
10. $a=69° 34' 56''$
    $c=120° 30' 30''$
    $\gamma=50° 10' 10''$.
11. $a=99° 40' 48''$
    $b=64° 23' 15''$
    $\alpha=95° 38' 4''$.
12. $a=50° 45' 20''$
    $b=69° 12' 40''$
    $\alpha=44° 22' 10''$.
13. $\alpha=110° 10'$
    $\beta=133° 18'$
    $a=147° 5' 32''$.
14. $\alpha=113° 39' 21''$
    $\beta=123° 40' 18''$
    $a=65° 39' 46''$.
15. $\alpha=100° 2' 11''$
    $\beta=98° 30' 28''$
    $a=95° 20' 39''$.
16. $\alpha=24° 33' 9''$
    $\beta=38° 0' 12''$
    $a=65° 20' 13''$.
17. $\alpha=117° 54.4'$
    $\beta=45° 8.6'$
    $a=76° 37.5'$.
18. $\alpha=104° 40'$
    $\beta=80° 13.6'$
    $a=126° 50.4'$.
19. $\alpha=115° 36' 45''$
    $\beta=80° 19' 12''$
    $b=84° 21' 56''$.
20. $\alpha=53° 18' 20''$
    $\beta=46° 15' 15''$
    $a=79° 30' 45''$.
21. $\alpha=61° 37' 53''$
    $\beta=139° 54' 34''$
    $b=150° 17' 26''$
22. $\alpha=70°$
    $\beta=120°$
    $b=80°$.
23. $\alpha=97° 20' 32''$
    $\beta=66° 57' 4''$
    $a=75° 0' 51''$.
24. $\alpha=51° 30'$
    $\beta=59° 16'$
    $a=63° 50'$.

CHAPTER VI

Applications and Problems

48. *Introduction.*—In Art. 2 the general scope of spherical trigonometry has been indicated and it has been pointed out that most of its uses and applications are in the scientific and technical subjects of geography, geodesy, navigation, and astronomy. There are also other important and interesting applications in solid geometry, and even in the highly technical subjects just mentioned there are many important problems which may be handled without special technical knowledge and are easily understood as soon as they are stated in spherical terminology. In this chapter we consider a few of these problems and applications.

49. *Surface and Volume of a Parallelepiped.*—A *parallelepiped* is a solid geometrical figure bounded by six plane faces which are parallelograms and which are arranged in three pairs of opposite equal and parallel faces. If all the dihedral angles formed by pairs of adjacent faces are right angles then the three edges which meet at each corner (or *vertex*) are mutually perpendicular and the figure is a *rectangular* parallelepiped. Thus an ordinary brick is a rectangular parallelepiped. If the dihedral angles at the edges and the trihedral angles at the corners are not all right angles the parallelepiped is *oblique*.

As shown in arithmetic or solid geometry *the volume of a rectangular parallelepiped is the product of the lengths of the three edges* which meet at one corner. The total surface of a rectangular parallelepiped is the sum of the areas of the six rectangular faces and is *twice the sum of the three products of pairs of the lengths of the three edges* which meet at a corner.

The rules or formulas which express the surface and volume of an *oblique parallelepiped* are more complicated and are now to be derived.

Fig. 20 represents an oblique parallelepiped with the eight vertices O, C, F, A; B, E, G, H and the twelve edges OC, CF, FA, AO; BE, EG, GH, HB; and OB, CE, FG, AH. The six faces are the rectangles formed by these edges.

The face $OCEB$ is taken as base; it is equal and parallel to the opposite face $AFGH$; D is a point in the base, and AD is perpen-

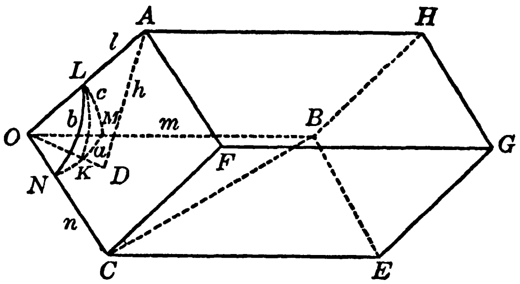


FIG. 20.

dicular to the plane of the base; $\overline{AD}=h$ is the *altitude* of the parallelepiped. $\triangle ODA$ is therefore a plane right triangle in a plane perpendicular to the plane of the base.

Let O be the center of a sphere of radius $\overline{OL}=\overline{OM}=\overline{ON}$; then LMN is a spherical triangle on the surface of this sphere intercepted by the trihedral angle $O\text{-}LMN$, and LKM, LKN are spherical triangles into which $\triangle LMN$ is divided by the plane ODA.

Let the three unequal edges of the parallelepiped be $\overline{OA}=l$, $\overline{OB}=m$, $\overline{OC}=n$. Let the face angles of the trihedral angle $O\text{-}ABC=O\text{-}LMN$ be $\angle BOC=a$, $\angle AOC=b$, $\angle AOB=c$; these are the angular measures of the sides of the spherical $\triangle LMN$, viz., $\overset{\frown}{MN}=a$, $\overset{\frown}{LN}=b$, $\overset{\frown}{LM}=c$.

The diagonal BC divides the base face $OCEB$ into the two equal plane triangles OBC, ECB and the area of each of these is $\triangle OBC = \frac{1}{2}mn \sin a$ [by (35), Art. 9]. The base area is therefore

$$[OCEB] = 2 \cdot (\triangle OBC) = mn \sin a. \tag{1}$$

Similarly $\qquad [OCFA] = ln \sin b,$

and $\qquad [OBHA] = lm \sin c.$

As each of these three faces is equal to its opposite face, the total surface area of the parallelepiped is

$$S = 2(mn \sin a + ln \sin b + lm \sin c). \tag{2}$$

Since the plane $ODA \perp OCEB$, the spherical $\triangle LKM$ is a right triangle with the right angle at K : $\angle LKM = 90°$. Then by (7) or (8), Art. 28,

$$\sin \widehat{KL} = \sin \widehat{LM} \sin \angle LMK$$

$$= \sin c \sin \angle LMK. \tag{3}$$

But in $\triangle LMN$ the three sides a, b, c are known, and also $\sin \angle LMK = 2 \sin \frac{1}{2}\angle LMK \cdot \cos \frac{1}{2}\angle LMK$. Using formulas (12), (13) of Art. 21, therefore:

$$\sin \angle LMK = \frac{2}{\sin a \sin c} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

This value of $\sin \angle LMK$ in (3) gives

$$\sin \widehat{KL} = \sin \angle DOA$$

$$= \frac{2}{\sin a} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}. \tag{4}$$

It is proved in solid geometry that the volume V of the parallelepiped is equal to the product (base area) $\times$ (altitude). Therefore

in Fig. 20, $V=[OCEB]\cdot h$. The area $[OCEB]$ is given by (1) above, and in plane rt. $\triangle ODA$, $h=l \sin \angle DOA$.

$$\therefore \quad V=lmn \sin a \sin \angle DOA.$$

Substituting $\sin \angle DOA$ from (4) in this, we have finally

$$V=2\, lmn \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}, \qquad (5)$$

where $s=\frac{1}{2}(a+b+c)$.

50. *Properties of Regular Polyhedrons.*—A *polyhedron* is a geometrical solid with any number (more than three) of plane surfaces which are bounded by straight lines. Thus the parallelepiped (Art. 49) is a polyhedron of six faces. Polyhedrons are named from the number of faces which they have; a few of these are as follows:

Number of Faces	*Name*
Four	Tetrahedron
Six	Hexahedron
Eight	Octahedron
Ten	Decahedron
Twelve	Dodecahedron
Twenty	Icosahedron

Thus a parallelepiped is a particular hexahedron.

The *faces* of a polyhedron are polygons of various numbers of sides. Thus the faces of the parallelepiped are parallelograms. These polygonal faces meet at the *edges* of the polyhedron, each adjacent pair of faces forming a dihedral angle; and at the *vertices* of the polyhedron, where they form polyhedral solid angles. It is proved in solid geometry that every polyhedron has the following remarkable property: *The number of faces plus the number of vertices equals the number of edges plus two.*

If all the faces of a polyhedron are *equal* (similar and congruent) *regular polygons* it is called a *regular polyhedron.* Thus a parallele-

piped, all of whose faces are equal squares (a cube), is a regular hexahedron. As only certain numbers of faces of certain shapes can be fitted together to form regular polyhedrons it is apparent that not all regular polyhedrons can be formed but only certain ones. It is in fact proved in solid geometry that *Only five regular polyhedrons are possible.*

The five regular polyhedrons are the following:

Faces	*Names*
Four equilateral triangles	Tetrahedron (triangular pyramid)
Six squares	Hexahedron (cube)
Eight equilateral triangles	Octahedron
Twelve regular pentagons	Dodecahedron
Twenty equilateral triangles	Icosahedron

As the faces of a regular polyhedron are regular polygons, each face may be divided into isosceles triangles by drawing lines from the center of the face to each vertex of the face. The number of these triangles is equal to the number of sides which the face has. Thus in Fig. 21 the regular pentagon $ABCDE$ is divided into the five equal isosceles $\triangle AFB$, BFC, CFD, DFE, EFA. $\triangle AFB$ is the type triangle for the pentagon, and its altitude $FG \perp AB$. $\angle AFB = \dfrac{360°}{5}$, and $\angle AFG = \frac{1}{2}\angle AFB = \dfrac{180°}{5}$. If s is the number of sides

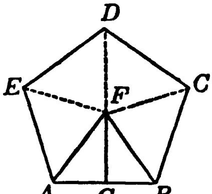


Fig. 21.

then $\angle AFG = \dfrac{180°}{s} = \dfrac{\pi}{s}$ rad.

In Fig. 22, $\triangle ACB$, ADB are two isosceles triangles such as $\triangle AFB$ in Fig. 21, which are parts of two adjacent faces of a regular polyhedron which meet in their common side AB, and AB is one of the equal edges of the polyhedron. CE, DE are apothems

of the polygonal faces of the polyhedron and are altitudes of the two triangles. Thus $CE \perp AB$ and $DE \perp AB$.

C and D are the centers of the two adjacent faces of the polyhedron (as F in Fig. 21); CO, DO are each perpendicular to the corresponding face, and $CO = DO$. O is the center of the polyhedron and OA, OB, joining the center to the vertices A, B of the polyhedron, are called *radii* of the polyhedron. $OE \perp AB$ at E the midpoint of AB, and OCE, ODE are right triangles in the plane $OCEDO$. $\angle CED$ is the dihedral angle of inclination of the two adjacent faces, and is the same for any two adjacent faces of the polyhedron. $\angle CED$ is called the *edge angle* of the polyhedron and is denoted by E.

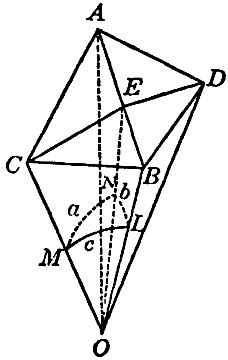


FIG. 22.

Let N = number of faces of polyhedron,
n = number of faces meeting at a vertex,
s = number of sides of each polygonal face,
a = length of each edge AB,
E = edge angle CED.

Describe a sphere about O as center, with radius $\overline{OL} = \overline{OM} = \overline{ON}$. The three face planes BOC, COE, EOB of the trihedral angle O-BCE intercept the *spherical triangle* LMN on the surface of this sphere. Let the angular measure of the sides of $\triangle LMN$ be

$$\overset{\frown}{LM} = c, \quad \overset{\frown}{MN} = a, \quad \overset{\frown}{NL} = b.$$

Then since plane $COE \perp$ plane EOB, $\angle LNM = 90°$, and spherical $\triangle LMN$ is a right triangle.

By Napier's Rules (Art. 29), in the spherical right $\triangle LMN$,

$$\cos \angle MLN = \cos a \sin \angle LMN. \tag{5a}$$

But since n faces of the polyhedron meet at B the dihedral angle

between planes COB, BOD through O is $\frac{360°}{n}$ and hence $\angle MLN = \frac{1}{2}\left(\frac{360°}{n}\right) = \frac{180°}{n} = \frac{\pi}{n}$ radians. Also, as in Fig. 22, $\angle BCA = \frac{\pi}{s} = \angle LMN$.

These values in (5*a*) give

$$\cos\frac{\pi}{n} = \cos a \cos\frac{\pi}{s}.$$

But
$$\begin{aligned}\cos a &= \cos \angle COE = \cos (90° - \angle CEO)\\ &= \sin \angle CEO\\ &= \sin \tfrac{1}{2}\angle CED = \sin\frac{E}{2}.\end{aligned}$$

$$\therefore\ \cos\frac{\pi}{n} = \sin\frac{E}{2}\sin\frac{\pi}{s};$$

$$\therefore\ \sin\frac{E}{2} = \cos\frac{\pi}{n}\csc\frac{\pi}{s}. \tag{6}$$

By means of this formula the edge angle of any regular polygon can be found when the number of sides of its face and the number of faces meeting at a vertex are known.

51. *Surface Area and Volume of a Regular Polyhedron.*—In the regular polygon of Fig. 21 the area of isosceles $\triangle AFB = \frac{1}{2}\overline{AB}\cdot\overline{FG} = \overline{AG}\cdot\overline{FG}$, and $\overline{FG} = \overline{AG}\cdot\cot\angle AFG = \overline{AG}\cdot\cot\frac{1}{2}\angle AFB$. Therefore area $\triangle AFB = \overline{AG}^2\cdot\cot\frac{1}{2}\angle AFB$. Similarly, in Fig. 22, in the regular face polygon, area $\triangle BEC = \overline{BE}^2\cdot\cot\frac{1}{2}\angle BCA$. But $\overline{BE} = \frac{1}{2}\overline{AB} = \frac{a}{2}$, and as already seen $\frac{1}{2}\angle BCA = \frac{\pi}{s}$. Therefore

$$\triangle BEC = \frac{a^2}{4}\cot\frac{\pi}{s}.$$

Now there are s triangles each equal to $\triangle BEC$ in one face of s sides, and hence the area of one face is (see also (50), Art. 10)

$$K = \frac{a^2}{4} s \cot \frac{\pi}{s}. \tag{7}$$

Also the regular polyhedron has N of these faces, all equal. The total surface area S is therefore

$$S = \tfrac{1}{4} N a^2 s \cot \frac{\pi}{s}. \tag{8}$$

Each face of the polyhedron is the base of a pyramid whose vertex is the center O in Fig. 22 and whose altitude is $\overline{CO} = \overline{DO}$, and it is proved in solid geometry that the volume is equal to $\frac{1}{3}$(altitude)$\times$(base area). When K is the area of one face, therefore, the volume of the pyramid is

$$V_1 = \tfrac{1}{3} \overline{CO} \cdot K. \tag{9}$$

Now in plane right $\triangle ECO$,

$$\begin{aligned} \overline{CO} &= \overline{CE} \cdot \tan \angle CEO \\ &= \overline{CE} \cdot \tan \tfrac{1}{2} \angle CED = \overline{CE} \cdot \tan \frac{E}{2}, \end{aligned}$$

and in right $\triangle CEB$, $\overline{CE} = \dfrac{a}{2} \cot \dfrac{\pi}{s}$. Therefore $\overline{CO} = \dfrac{a}{2} \cot \dfrac{\pi}{s} \tan \dfrac{E}{2}$.

This in (9) gives

$$V_1 = \left(\frac{a}{6} \cot \frac{\pi}{s} \tan \frac{E}{2} \right) \cdot K,$$

and the volume V of the polyhedron consists of N such pyramids. Multiplying V_1 by N, therefore, and substituting the value of K given by (7) above,

$$V = NV_1 = N \left(\frac{a}{6} \cot \frac{\pi}{s} \tan \frac{E}{2} \right) \cdot \frac{a^2}{4} s \cot \frac{\pi}{s}.$$

$$\therefore \quad V = \frac{Na^3 s}{24} \cot^2 \frac{\pi}{s} \tan \frac{E}{2}. \tag{10}$$

To compute the volume of a regular polyhedron by this formula, the edge angle $E\left(\text{or } \frac{E}{2}\right)$ is first found by means of formula (6),

$$\sin\frac{E}{2} = \cos\frac{\pi}{n}\csc\frac{\pi}{s}, \tag{6}$$

and this value of $\frac{E}{2}$ is then used in the volume formula (10).

Exercises

1. Show by means of formula (2), Art. 49, that the surface of a rectangular parallelepiped is $S=2\ (lm+mn+nl)$.
2. Show by means of (5) that the volume of a rectangular parallelepiped is $V=lmn$.
3. If e is the edge of a cube show that formulas (2) and (5) lead to $S=6e^2$ and $V=e^3$.
4. If all the edges of a rhombic parallelepiped are equal to e and each of the three face angles of its smallest trihedral corner angle is 30° find the total surface.
5. Find the volume of the solid in Ex. 4.
6. The three unequal edges of a parallelepiped are 1, 2, 3 units and the three opposite face angles at its bluntest corner are 75°, 90°, 105°. Find the total surface.
7. Find the volume of the solid in Ex. 6.

The five regular polyhedrons are described in the table in Art. 50. By means of formula (6) in Art. 50 find the dihedral edge angle of each, as follows:

8. Tetrahedron (triangular pyramid), three faces meeting at each vertex ($n=3$).
9. Hexahedron (cube, $n=3$).
10. Octahedron ($n=4$).
11. Dodecahedron ($n=3$).
12. Icosahedron ($n=5$).

Find the exact (not decimal) value of $\cos E$ for each of the regular polyhedrons as follows: (Hint, use formula (6) above and $\cos E = 1-2\sin^2\frac{E}{2}$ from (13), Art. 7.)

13. Tetrahedron.
14. Cube.
15. Octahedron.
16. Dodecahedron.
17. Icosahedron.

Find the total surface of each of the regular polyhedrons in terms of the edge a as follows:

18. Tetrahedron.
19. Cube.
20. Octahedron.
21. Dodecahedron.
22. Icosahedron.

Find the volume of each of the regular polyhedrons in terms of the edge a as follows:

23. Tetrahedron.
24. Cube.
25. Octahedron.
26. Dodecahedron.
27. Icosahedron.

28. Show that the radius $r = \overline{OC}$ of the sphere inscribed in the regular polyhedron of Fig. 22 is equal to $\frac{a}{2} \cot \frac{\pi}{s} \tan \frac{E}{2}$.

52. *The Earth or Terrestrial Sphere.*—The earth is one of the nine planets which revolve around the sun, each of which requires a certain particular time to complete each revolution. The time of revolution of the earth about the sun is the *year*. The path followed by each planet in each revolution is called its *orbit*. The earth's orbit lies in one plane, called its *orbital plane*, and is very nearly a circle with the sun at its center. More exactly, the orbit is very slightly elongated or *elliptical* and the sun is at a *focus* of the ellipse.

The earth itself is globe-shaped, very nearly an exact sphere. The slight variation from a sphere is detected only by very precise measurements, and in all ordinary measurements and calculations may be disregarded. The surface of the earth is also not perfectly smooth, but when the earth is represented by a circle of about six inches diameter drawn with an ordinary pencil or by a circle of about six feet diameter drawn with ordinary blackboard crayon the deviation and irregularities lie within the pencil or chalk line. For all ordinary purposes the earth is therefore considered as a sphere. It is called the *terrestrial sphere* in distinction from the celestial sphere, described in Art. 58.

The average radius of the earth is very nearly 6367.65 kilo-

meters. In United States land miles, therefore, the radius, diameter, and circumference of the terrestrial sphere are very nearly

$$\left.\begin{aligned} R &= 3956.8, \\ D = 2R &= 7913.5, \\ C = \pi D &= 24{,}861 \text{ miles.} \end{aligned}\right\} \tag{11}$$

In approximate calculations these values are often taken as about 4000, 8000, 25,000 land miles, respectively.

The circumference of the spherical earth is the circumference of a *great circle* on its surface. As the circle contains 360 degrees, an arc of one degree on this great circle has a length of $\frac{24{,}861}{360} =$ 69.06 or about 69.1 land miles.

The length of *one minute* of arc on the great circle of the earth is called the *nautical mile*. Since the circle contains $60 \times 360 = 21{,}600$ minutes there are 21,600 nautical miles in the circumference, and as this circumference is 24,861 land miles

$$1 \text{ nautical mile} = \frac{24{,}861}{21{,}600} = 1.151 \text{ land miles,}$$

$$1 \text{ land mile} = \frac{21{,}600}{24{,}861} = 0.8688 \text{ nautical mile.}$$

As the earth moves in its orbit about the sun it also *rotates* once in each 24 hours about a certain fixed diameter, as an axis. This *axis* of the earth remains fixed always in the same direction and is inclined to the orbital plane at an angle of $66° \ 32' \ 52''$, or very nearly $66° \ 33'$.

The ends of the axis diameter of the earth are called the *poles* of the earth. The great circle which has the same points as poles is called the *equator* of the earth. The equator is therefore the intersection of the earth's surface with a plane through the center

and perpendicular to the axis. This plane is called the *equatorial plane*.

If an observer stands on the equator and faces toward the rising sun one of the earth's poles is to his left and one to his right. The pole to the left is called the *north* pole and the other the *south* pole. Alternatively, the direction of the rising sun is called the *east* and that of the setting sun the *west;* and if the equatorial observer

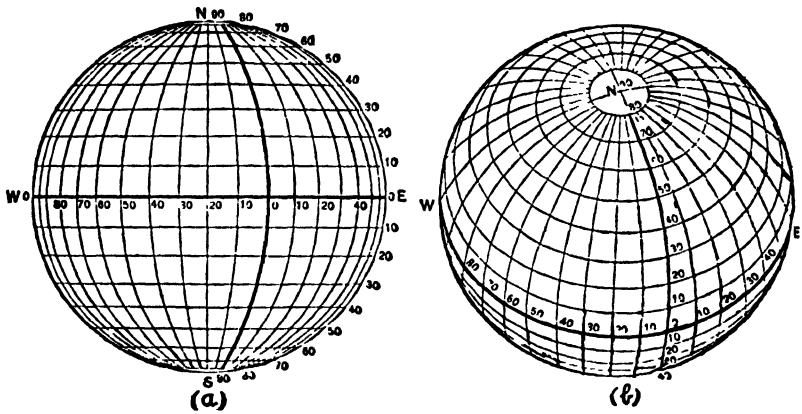


FIG. 23.

stands with the east to his right and the west to his left, he then faces toward the north and the south is at his back. The direction of rotation of the earth about its axis is *toward* the rising sun, that is *from west to east.*

In Fig. 23, the terrestrial sphere is shown with the north and south poles marked N, S and the east and west directions on the equator as E, W, respectively. Small circles on the surface, which are the intersections of planes parallel to the equatorial plane, are called *parallels of latitude*, and great circles through the poles and perpendicular to the equator are called *meridians of longitude.*

Fig. 23(a) shows the sphere (earth) as it appears to an observer situated in the equatorial plane outside the earth, and (b) shows

the sphere as it appears to an observer outside the earth and some distance from the equatorial plane toward the north.

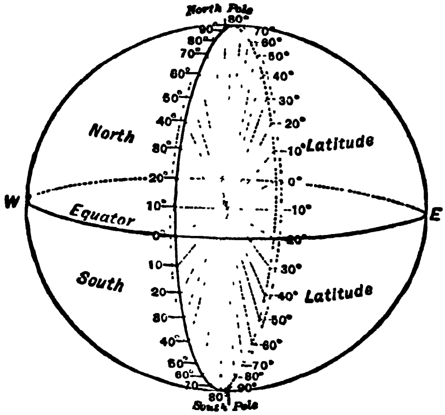
The method of locating and denoting the parallels of latitude and meridians of longitude is shown in Fig. 23, and in more detail in Fig. 24. As the equator and poles divide any meridian circle into quadrants, latitude is measured along any meridian from the equator toward each pole, and any parallel is denoted by giving the meridian arc measure in degrees from the equator to that parallel, with the specification N or S, or correspondingly $+$ or $-$, for north or south, respectively. This arc measure is called the *latitude* of the parallel and is equal to the plane angle at the center of the sphere (earth) between the radii drawn to the ends of the arc on the meridian circle, as shown in Fig. 24(a). The angle between the axis and the radius to any parallel is the complement of the latitude and is called the *co-latitude* of that parallel, or sometimes the *polar distance* of the parallel.

The latitude of a standard reference marker in New York City is $40° 46' 47.17'' N$ or $+40° 46' 47.17''$.

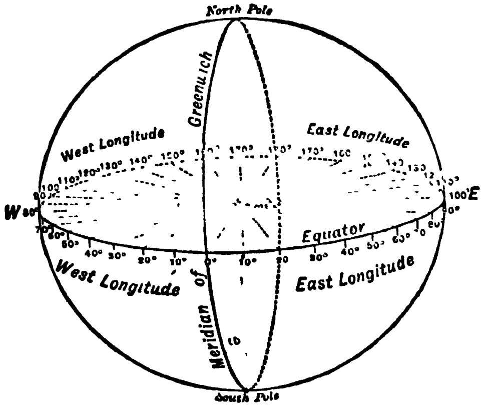
The great circle on which longitude is measured is the equator. There is no natural dividing line, as the equator is for latitude, so an arbitrary line is chosen, and by international agreement the meridian passing through the British Royal Observatory at Greenwich, England, near London, is taken as the reference circle or *prime meridian*, as shown in Fig. 24(b).

The Greenwich or prime meridian divides the equator into two semi-circles and longitude is measured west (W or $+$) and east (E or $-$) from this meridian through 180° along the equator to the point where this meridian crosses the equator on the opposite side of the earth. The westward direction is chosen as positive because the apparent daily motion of the sun is toward the west.

The *longitude* of any meridian is then the measure in degrees of the smaller arc of the equator between that meridian and the prime meridian. It is also the smaller plane angle between the


North Pole
North
Latitude
W
Equator
E
South
Latitude
South Pole

(a)


North Pole
Greenwich
West Longitude
East Longitude
W
E
Equator
West Longitude
East Longitude
Meridian of
South Pole

(b)

FIG. 24.

radii to the intersections of the equator with the prime meridian and the given meridian; and it is the smaller spherical angle at either pole between those two meridians. These angles are shown in Fig. 24(*b*) and also in Fig. 23, and the longitude of any meridian is less than 180°. The longitude of the marker in New York City mentioned above is 73° 58′ 41.00″ *W* or + 73° 58′ 41.00″.

The longitude of a *point* on the earth's surface is the longitude of the meridian passing through it, and the latitude of such a point is the latitude of the parallel passing through it. Any such point is completely located by stating its latitude and longitude. Thus the marker referred to above is at 40° 46′ 47.17″ *N*, 73° 58′ 41.00″ *W* (+40° 46′ 47.17″, +73° 58′ 41.00″).

The *difference of latitude* of two points on the earth's surface is the arc of a meridian included between the parallels of the two points. It is equal to the algebraic difference obtained by subtracting the smaller of two latitudes of the same sign, or the negative if of opposite signs, from the other.

The *difference of longitude* of two points on the earth's surface is the smaller arc of the equator included between the meridians of the two points; it is equal to the smaller spherical angle at the pole between the two meridians. It is the algebraic difference obtained by subtracting the smaller of two longitudes of the same sign, or the negative if of opposite signs, from the other. If this algebraic difference is numerically greater than 180°, the numerical difference between it and 360° is used.

The general science of the form and size of the earth and of the consequences of its axial rotation and orbital revolution, so far as they are related to terrestrial matters, is called *mathematical geography*. The science of the measurement of its form and size is called *geodesy*.

The measurements and calculations which are required to determine the latitude and longitude of any point on the earth's surface are explained in books on *spherical and practical astronomy* or on *navigation and nautical astronomy*.

53. *Shortest Line Joining Two Points on the Earth's Surface.*— It is proved in solid geometry that the shortest line which can be drawn on the surface of a sphere to join two points on the surface is the lesser of the two arcs of the *great circle* which passes through the two points. The measure of this arc in angle or length units is called the *distance* from one of the points to the other.

If the distance is measured in *minutes* of arc this is also the distance in *nautical miles*, and this number multiplied by 1.151 is the distance in *land miles*.

If the distance is measured in *radians* of arc the distance in land miles is the number of radians multiplied by the radius $R = 3956.8$.

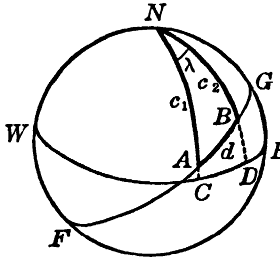


FIG. 25.

To find the distance when the two points are given by their latitudes and longitudes requires the solution of a spherical triangle for one side. In Fig. 25 let N be the north pole and WE the equator; and let A, B be the given points, and FG the great circle passing through A and B. Then $\overset{\frown}{AB} = d$ is the required distance. $\overset{\frown}{CA} = L_1$ and $\overset{\frown}{DB} = L_2$ are the latitudes of A and B; $\overset{\frown}{NA} = c_1 = 90° - L_1$, $\overset{\frown}{NB} = c_2 = 90° - L_2$ are their co-latitudes; and $\overset{\frown}{CD} = \angle CND = \lambda = \lambda_1 - \lambda_2$ is the difference of longitude of the given points A, B.

When the two points are given by their latitudes and longitudes, therefore, the co-latitudes c_1, c_2 and the difference of longitude λ are known. In the spherical triangle ABN, therefore, *two sides and the included angle* are known, and the *third side is required.* The complete solution of the triangle is therefore not required, and the method of Art. 44 is to be used.

The solution formula is (17) of Art. 44; with the notation of Fig. 25 it is

$$\text{hav } d = \text{hav } (c_1 - c_2) + \sin c_1 \sin c_2 \text{ hav } \lambda. \tag{12}$$

But $\qquad c_1 - c_2 = (90° - L_1) - (90° - L_2) = L_2 - L_1,$

$\qquad \sin c_1 = \sin (90° - L_1) = \cos L_1,$

$\qquad \sin c_2 = \sin (90° - L_2) = \cos L_2,$

and $\qquad \lambda = \lambda_2 - \lambda_1$ or $\lambda_1 - \lambda_2$ (as in Art. 52),

when $\qquad \lambda_1, \lambda_2 =$ longitudes of A, B.

Substituting these values in (12), the solution formula becomes

$$\text{hav } d = \text{hav } (L_2 - L_1) + \cos L_1 \cos L_2 \text{ hav } (\lambda_2 - \lambda_1). \tag{13}$$

When the points A, B are given by their latitudes and longitudes thus: A (L_1, λ_1) and B (L_2, λ_2), formula (13) gives the distance d in angular measure at once by the computation illustrated in Art. 44. The quantity Q is here, however,

$$Q = \cos L_1 \cos L_2 \text{ hav } (\lambda_2 - \lambda_1),$$

using cosines instead of sines. The distance d is expressed in miles as explained above.

The *difference of latitude* $L_2 - L_1$ and the *difference of longitude* $\lambda_2 - \lambda_1$ are found as explained in Art. 52.

54. *Great Circle Sailing.*—The shortest route between two seaports is the great circle arc joining them, as in Art. 53, and unless the distance is very short or there are obstacles on the route the great circle route is generally followed in sailing a ship from one port to another.

In the language of *navigation* the route or path followed by a ship is called the ship's *track*, the spherical angle which the track makes with the meridian at any point is the *course* at that point, and the point on the great circle track which is nearest to the pole is the *vertex* of the track. As a great circle track crosses various meridians and parallels at various angles, the vertex is on the

parallel farthest from the equator and has the greatest latitude or smallest co-latitude of any point on the track.

The procedure and operations necessary to direct a ship along a great circle track are called *great circle sailing*, and when a navigator determines the course of the ship's track at any point he is said to *lay a course* from that point.

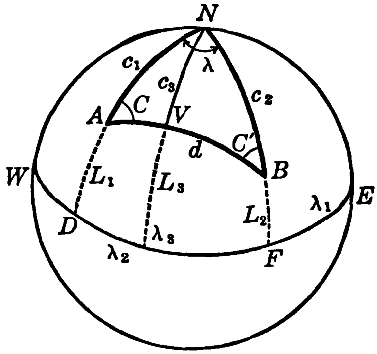


FIG. 26.

In Fig. 26, N is the north pole and WE is the equator; A, B are two points on a great circle track, and $\widehat{AB}$ is the great circle arc joining them. If the ship is to sail from A to B the course C at A is called the *first course* and C' at B the *last course*, and vice versa. If the ship is to sail either way or both ways, C is called the *first course from A* and C' the *first course from B*.

The point V is the vertex of the track and $\widehat{NV} \perp \widehat{AB}$.

In the great circle sailing on the track $\widehat{AB}$ the points A, B are given by their latitudes L and longitudes λ: $A(L_1, \lambda_1)$, $B(L_2, \lambda_2)$, and it is required to find the courses C, C'; the distance $\widehat{AB} = d$; and the position (latitude and longitude) of the vertex V.

With $A(L_1, \lambda_1)$ and $B(L_2, \lambda_2)$ given, their co-latitudes c_1, c_2 and their difference of longitude λ are known, and to find C, C', d requires the *complete solution* of the spherical $\triangle ABN$ with *two sides and included angle* given, Art. 43.

If in Fig. 26 NE is the prime meridian (it is not necessarily so),

$$L_1 = \widehat{DA},\ L_2 = \widehat{FB} \text{ are the given latitudes,}$$

$$\lambda_1 = \widehat{EF},\ \lambda_2 = \widehat{ED} \text{ are the given longitudes,}$$

$\lambda = \widehat{DE} = \lambda_1 - \lambda_2 = \angle ANB$ is the difference of longitude,
$c_1 = 90° - L_1$, $c_2 = 90° - L_2$ are the co-latitudes,
$C = \angle BAN$, $C' = \angle ABN$ are the courses,
d, V = distance and vertex of the track.

Using this notation, the solution formulas of Art. 43 are

$$\tan \tfrac{1}{2}(C+C') = \frac{\cot \frac{\lambda}{2} \cos \frac{1}{2}(c_2 - c_1)}{\cos \frac{1}{2}(c_2 + c_1)},$$

$$\tan \tfrac{1}{2}(C-C') = \frac{\cot \frac{\lambda}{2} \sin \frac{1}{2}(c_2 - c_1)}{\sin \frac{1}{2}(c_2 + c_1)},$$

$$C,\ C' = \tfrac{1}{2}(C+C') \pm \tfrac{1}{2}(C-C'),$$

$$\tan \frac{d}{2} = \frac{\tan \frac{1}{2}(c_2 - c_1) \sin \frac{1}{2}(C+C')}{\sin \frac{1}{2}(C-C')}.$$

But $\quad c_2 - c_1 = (90° - L_2) - (90° - L_1) = L_1 - L_2,$

$$c_2 + c_1 = (90° - L_2) + (90° - L_1) = 180° - (L_1 + L_2).$$

$$\therefore \ \cos \tfrac{1}{2}(c_2 + c_1) = \cos [90° - \tfrac{1}{2}(L_1 + L_2)],$$
$$= \sin \tfrac{1}{2}(L_1 + L_2)$$
$$\sin \tfrac{1}{2}(c_2 + c_1) = \cos \tfrac{1}{2}(L_1 + L_2),$$

and these values in the solution formulas give

$$\left.\begin{aligned} \tan \tfrac{1}{2}(C+C') &= \frac{\cot \frac{1}{2}(\lambda_1 - \lambda_2) \cos \frac{1}{2}(L_1 - L_2)}{\sin \frac{1}{2}(L_1 + L_2)}, \\ \tan \tfrac{1}{2}(C-C') &= \frac{\cot \frac{1}{2}(\lambda_1 - \lambda_2) \sin \frac{1}{2}(L_1 - L_2)}{\cos \frac{1}{2}(L_1 + L_2)}, \\ C,\ C' &= \tfrac{1}{2}(C+C') \pm \tfrac{1}{2}(C-C'), \\ \tan \frac{d}{2} &= \frac{\tan \frac{1}{2}(L_1 - L_2) \sin \frac{1}{2}(C+C')}{\sin \frac{1}{2}(C-C')}. \end{aligned}\right\} \qquad (14)$$

When the given latitudes L and longitudes λ are used directly in these formulas three logarithmic computations give the courses C, C' and distance d in angular measure immediately. The distance is then expressed in miles as already explained.

The vertex of the ship's track V is now to be located, that is, the latitude L_3 and longitude λ_3 are to be determined. These are found from the spherical right triangle AVN, right-angled at V.

In rt.$\triangle AVN$ the co-latitude $\overset{\frown}{NA}=c_1$ and the course $\angle NAV=C$ are now known, and when the side $\overset{\frown}{NV}=c_3=90°-L_3$ and the acute $\angle ANV=\lambda_2-\lambda_3$ are found, $V(L_3, \lambda_3)$ is then completely located.

In the right $\triangle AVN$ the *hypotenuse* c_1 *and angle* C are given and the method of Art. 31 is used. In the notation of Fig. 26 the solution formulas are

$$\sin c_3=\sin c_1 \sin C,$$

$$\cot \angle AVN=\cos c_1 \tan C.$$

But $c_1=90°-L_1$, $c_3=90°-L_3$; therefore $\sin c_3=\cos L_3$, $\sin c_1=\cos L_1$, $\cos c_1=\sin L_1$, and the formulas become

$$\left.\begin{aligned} \cos L_3&=\cos L_1 \cos C,\\ \cot \angle AVN&=\sin L_1 \tan C,\\ \lambda_3&=\lambda_1-\angle AVN. \end{aligned}\right\} \tag{15}$$

When the given value of L_1 and the value of C found by (14) are used, two computations give $V(L_3, \lambda_3)$.

55. *Legendre's Theorem.*—We now develop an important theorem known as *Legendre's theorem* which is very useful in connection with measurements made on the earth's surface.

In any spherical triangle with interior angles α, β, γ, let the *lengths* of the opposite sides (in linear measure) be a, b, c. The *angular* measure of the sides (in radians) is then, by (1), Art. 5,

$\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$ when r is the radius of the sphere. The first of the fundamental formulas (3) of Art. 17, in which the sides must be expressed in angular measure, is then

$$\cos\frac{a}{r}=\cos\frac{b}{r}\cos\frac{c}{r}+\sin\frac{b}{r}\sin\frac{c}{r}\cos\alpha.$$

$$\therefore\ \cos\alpha=\frac{\cos\frac{a}{r}-\cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}}, \tag{16}$$

and similarly for $\cos\beta$ and $\cos\gamma$.

According to (18), Art. 7, in which the angle θ in radian measure may be $\frac{a}{r}, \frac{b}{r}$, or $\frac{c}{r}$,

$$\cos\frac{a}{r}=1-\frac{a^2}{2r^2}+\frac{a^4}{24r^4}-\frac{a^6}{720r^6}+\cdots,$$

$$\sin\frac{b}{r}=\frac{b}{r}-\frac{b^3}{6r^3}+\frac{b^5}{120r^5}-\frac{b^7}{5040r^7}+\cdots,$$

and similarly for $\cos\frac{b}{r}$, $\cos\frac{c}{r}$, $\sin\frac{c}{r}$. If the lengths a, b, c are very small as compared with r, the fractions $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$ are very small and powers of these fractions above the fourth may be disregarded in comparison with the lower powers. Omitting the higher powers in the series and substituting the resulting formulas for $\cos\frac{a}{r}$, $\sin\frac{b}{r}$, etc., in (16), we have, therefore,

$$\cos\alpha=\frac{\left(1-\frac{a^2}{2r^2}+\frac{a^4}{24r^4}\right)-\left(1-\frac{b^2}{2r^2}+\frac{b^4}{24r^4}\right)\left(1-\frac{c^2}{2r^2}+\frac{c^4}{24r^4}\right)}{\left(\frac{b}{r}-\frac{b^3}{6r^3}\right)\left(\frac{c}{r}-\frac{c^3}{6r^3}\right)}.$$

On carrying out the indicated multiplications, taking out the common factors $\frac{b}{r}$, $\frac{c}{r}$ in the denominator, and again omitting powers higher than the fourth,

$$\cos\alpha=\frac{\left(\frac{b^2+c^2-a^2}{2r^2}+\frac{a^4-b^4-c^4-6b^2c^2}{24r^4}\right)}{\frac{bc}{r^2}\left(1-\frac{b^2+c^2}{6r^2}\right)}$$

$$=\frac{r^2}{bc}\left(\frac{b^2+c^2-a^2}{2r^2}+\frac{a^4-b^4-c^4-6b^2c^2}{24r^4}\right)\left(1+\frac{b^2+c^2}{6r^2}\right),$$

since $\frac{1}{\left(1-\frac{b^2+c^2}{6r^2}\right)}=1+\frac{b^2+c^2}{6r^2}$ very nearly when $\frac{b^2+c^2}{6r^2}$ is small as compared with 1. Again carrying out indicated multiplications and omitting powers higher than the fourth, we get

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}-\frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{24bcr^2}. \tag{17}$$

Now if α', β', γ' are the angles of the *plane* triangle whose opposite sides have the *same lengths* a, b, c as those of the spherical triangle, we have from the first of the formulas (24) of Art. 9,

$$\cos\alpha'=\frac{b^2+c^2-a^2}{2bc}. \tag{18}$$

$$\therefore\ \sin^2\alpha'=1-\cos^2\alpha'=1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2c^2}$$

$$=\frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{4b^2c^2}.$$

$$\therefore\ \frac{bc\sin^2\alpha'}{6r^2}=\frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{24bcr^2} \tag{19}$$

Substituting (18) and (19) in (17),

$$\cos\alpha=\cos\alpha'-\frac{bc\sin^2\alpha'}{6r^2}, \tag{20}$$

and similarly for β, β' and γ, γ'.

This result applies to a *spherical* triangle with angles α, β, γ and sides whose lengths a, b, c are very small as compared with the radius r of the sphere; and to a *plane* triangle with angles α', β', γ' and sides of the *same lengths* a, b, c. The angles α', β', γ' differ but little from α, β, γ and therefore $\alpha-\alpha'=\phi$ is a small quantity. From this $\alpha=\alpha'+\phi$ and

$$\cos\alpha=\cos(\alpha'+\phi)=\cos\alpha'\cos\phi-\sin\alpha'\sin\phi.$$

But when ϕ is very small, then $\cos\phi=1$ and $\sin\phi=\phi$ very nearly. Therefore

$$\cos\alpha=\cos\alpha'-\phi\sin\alpha'$$

very nearly. Comparing this formula with the corresponding formula (20) it is seen that

$$\phi\sin\alpha'=\frac{bc\sin^2\alpha'}{6r^2}.$$

$$\therefore\quad \phi=\alpha-\alpha'=\frac{1}{3r^2}\cdot\tfrac{1}{2}bc\sin\alpha'. \tag{21}$$

Now by the second formula (35) of Art. 9, $\frac{1}{2}bc\sin\alpha'=K'$ is the area of the plane triangle with angles α', β', γ' and sides a, b, c. Formula (21) is therefore

Similarly

$$\left.\begin{aligned}\alpha-\alpha'&=\frac{K}{3r^2}.\\ \beta-\beta'&=\frac{K}{3r^2},\\ \gamma-\gamma'&=\frac{K}{3r^2}.\end{aligned}\right\} \tag{22}$$

Adding, $$\alpha+\beta+\gamma-(\alpha'+\beta'+\gamma')=3\left(\frac{K}{3r^2}\right)=\frac{K}{r^2}.$$

But in the plane triangle, $\alpha'+\beta'+\gamma'=180°=\pi$.

$$\therefore\ (\alpha+\beta+\gamma)-\pi=\frac{K}{r^2},$$

and the left member of this equation is the *spherical excess* E of the spherical triangle with angles α, β, γ. Therefore, for this triangle (whose sides are small as compared with the radius r),

$$E=\frac{K}{r^2},\quad \frac{K}{3r^2}=\frac{E}{3}.$$

The three formulas (22) become, therefore,

$$\alpha-\alpha'=\beta-\beta'=\gamma-\gamma'=\frac{E}{3}. \tag{23}$$

These three equations express in symbols

Legendre's Theorem.—*If the sides of a spherical triangle are very small compared with the radius of the sphere, each angle of the spherical triangle exceeds the corresponding angle of the plane triangle whose sides are of the same length, by an amount equal to one-third the spherical excess of the spherical triangle.*

The radius of the earth being nearly 4000 land miles, most land, highway, city street, and local map measurements on the terrestrial sphere involve spherical triangles whose sides are small compared with the radius, and therefore Legendre's theorem applies to such measurements. The angles of such a spherical triangle are much more easily measured than the sides, and hence the spherical excess and therefore the angles of the plane triangle with the same sides are known at once. The sides of the plane triangle are much more quickly and easily computed than those of the spherical triangle, and so if one side of the triangle is known the other two are found at once.

56. *An Application of Legendre's Theorem.*—In the New York State Survey the *angles* of a spherical triangle with vertices at the

towns of Howlett, Eagle, and Gilbertsville were measured. After making proper corrections and reductions these were found to be

$$\begin{cases} \text{At Howlett} & \alpha = 85^\circ\ 18'\ 56.962'' \\ \text{At Eagle} & \beta = 51^\circ\ 35'\ 42.965'' \\ \text{At Gilbertsville} & \gamma = 43^\circ\ \ 5'\ 22.749'' \end{cases}$$

$$\text{Adding,} \qquad \alpha+\beta+\gamma = 180^\circ\ 0'\ 2.676''$$

$$\text{Excess} \qquad E = \alpha+\beta+\gamma-180^\circ \;=\; 2.676''$$

$$\frac{E}{3} = \frac{2.676''}{3} \;=\; 0.892''$$

$$\alpha' = \alpha - \frac{E}{3} = 85^\circ\ 18'\ 56.962'' - .892'' = 85^\circ\ 18'\ 56.070''$$

$$\beta' = \beta - \frac{E}{3} = 51^\circ\ 35'\ 42.965'' - .892'' = 51^\circ\ 35'\ 42.073''$$

$$\gamma' = \gamma - \frac{E}{3} = 43^\circ\ \ 5'\ 22.749'' - .892'' = 43^\circ\ \ 5'\ 21.857''.$$

The angles α', β', γ' are the angles of the plane triangle whose sides a, b, c are the same as the required sides of the geodetic spherical triangle.

The side b of the geodetic triangle, the distance from the Howlett vertex to the Gilbertsville vertex, was already known in meters as the result of other measurements and computations in the survey. From those computations

$$\log b = 4.5422732.$$

The plane triangle (α', β', γ', a, b, c) is now solved at once by formulas (23) of Art. 9, and the computations give

$$\log a = 4.6467037,$$

$$\log c = 4.4826658.$$

Therefore the great circle *distances* of the three towns named are:

$$\begin{cases} \text{Eagle to Gilbertsville:} & a=44330.61 \text{ meters} \\ \text{Gilbertsville to Howlett:} & b=34855.66 \text{ meters} \\ \text{Howlett to Eagle:} & c=30385.46 \text{ meters} \end{cases}$$

57. *Area of a Small Terrestrial Triangle.*—Two other useful formulas or theorems are easily obtained from Legendre's theorem. Thus formula (32), Art. 26, for the spherical triangle is

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2}\tan\left(\frac{s-a}{2}\right)\tan\left(\frac{s-b}{2}\right)\tan\left(\frac{s-c}{2}\right)}$$

when the sides are expressed in angular measure. When a, b, c are the *lengths* of the sides the radian angular measures are $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$ as in Art. 55 and the formula for $\frac{E}{4}$ is

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2r}\tan\left(\frac{s-a}{2r}\right)\tan\left(\frac{s-b}{2r}\right)\tan\left(\frac{s-c}{2r}\right)}. \quad (24)$$

By the third of the series formulas (18) of Art. 7, neglecting powers higher than the fourth,

$$\tan\frac{s}{2r}=\frac{s}{2r}+\frac{\left(\frac{s}{2r}\right)^3}{3}=\frac{s}{2r}\left[1+\frac{s^2}{12r^2}\right],$$

$$\tan\left(\frac{s-a}{2r}\right)=\frac{s-a}{2r}+\frac{\left(\frac{s-a}{2r}\right)^3}{3}=\frac{s-a}{2r}\left[1+\frac{(s-a)^2}{12r^2}\right],$$

and similarly

$$\tan\left(\frac{s-b}{2r}\right)=\frac{s-b}{2r}\left[1-\frac{(s-b)^2}{12r^2}\right],\quad \tan\left(\frac{s-c}{2r}\right)=\frac{s-c}{2r}\left[1-\frac{(s-c)^2}{12r^2}\right].$$

Substituting these values in (24),

$$\tan\frac{E}{4}=\sqrt{\frac{s}{2r}\left(\frac{s-a}{2r}\right)\left(\frac{s-b}{2r}\right)\left(\frac{s-c}{2r}\right)\left[1+\frac{s^2}{12r^2}\right]\left[1+\frac{(s-a)^2}{12r^2}\right] \times\left[1+\frac{(s-b)^2}{12r^2}\right]\left[1+\frac{(s-c)^2}{12r^2}\right]}$$

$$=\frac{1}{4r^2}\sqrt{s(s-a)(s-b)(s-c)\left[1+\frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{12r^2}\right]}, \tag{25}$$

the quantity in square brackets under the last radical being obtained by multiplying the four quantities in square brackets under the first radical.

But when the excess E is small, $\frac{E}{4}$ is very small and $\tan\frac{E}{4}=\frac{E}{4}$ very nearly. Also by formula (38), Art. 8,

$$\sqrt{s(s-a)(s-b)(s-c)}=K'$$

is the area of the *plane* triangle with sides a, b, c. Therefore (25) becomes

$$\frac{E}{4}=\frac{K'}{4r^2}\sqrt{1+\frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{12r^2}}.$$

Now $s=\frac{1}{2}(a+b+c)$, $s-a=\frac{-a+b+c}{2}$, $s-b=\frac{a-b+c}{2}$, $s-c=\frac{a+b-c}{2}$, and these values substituted under the last radical give

$$\frac{E}{4}=\frac{K'}{4r^2}\sqrt{1+\frac{a^2+b^2+c^2}{12r^2}}=\frac{K'}{4r^2}\left[1+\left(\frac{a^2+b^2+c^2}{12r^2}\right)\right]^{\frac{1}{2}}.$$

Applying the Binomial Theorem of algebra to the last expression and omitting powers higher than the fourth,

$$\frac{E}{4}=\frac{K'}{4r^2}\left[1+\left(\frac{a^2+b^2+c^2}{24r^2}\right)\right]$$

$$\therefore\quad Er^2=K'+\left(\frac{a^2+b^2+c^2}{24r^2}\right)\cdot K'.$$

But by (6*b*), Art. 16, when r is the radius of the sphere and K is the area of the *spherical* triangle, $Er^2=K$.

$$\therefore\quad K=K'+\left(\frac{a^2+b^2+c^2}{24r^2}\right)\cdot K' \tag{26}$$

or

$$\frac{K-K'}{K'}=\frac{a^2+b^2+c^2}{24r^2}. \tag{27}$$

Formula (26) shows that *when the spherical and plane triangles have sides a, b, c, of the same length, small compared with the radius r, the area of the spherical triangle exceeds that of the plane triangle by the fraction $\frac{a^2+b^2+c^2}{24r^2}$ of the latter.*

As the area of a plane triangle is more readily calculated than that of a spherical triangle this result enables the spherical area to be found very readily from the plane area. In particular, *when a, b, c are very small compared with r,* $\frac{a^2+b^2+c^2}{24r^2}$ is much smaller, and (26) becomes very nearly $K=K'$; that is, *the plane and spherical areas are equal,* very nearly.

Exercises

1. Find the shortest airline distance from New York City (see Art. 52) to Rio de Janeiro (22° 54.4′ S, 43° 10.4′ W).
2. Find the shortest distance from New York City to Paris (48° 50.2′, 2° 20.2′ E).
3. Determine from a map or globe the latitudes and longitudes of any two

chosen cities or other points on the earth and find their distance in nautical and land miles.

4. In sailing from New York to Liverpool, Cunard liners normally lay their course from a point off New York at 42° *N*, 50° *W* to Fastnet Rock off the Irish coast at 51° 24′ *N*, 9° 36′ *W*. To follow the great circle track find the (*a*) first course; (*b*) vertex of the track; and (*c*) distance.

5. San Francisco is at 37° 48′ *N*, 122° 28′ *W*, and Yokohama, Japan, is at 36° 26′ *N*, 139° 39′ *E*. What is the shortest air distance joining the two cities?

6. Find the courses, vertex, and distance of the great circle track joining Vancouver Island (50° *N*, 128° *W*) and Honolulu (40° *N*, 74° *W*).

7. Find the great circle distance from *A*(54° 18′ *N*, 142° 38′ *E*) to *B*(7° 12′ *S*, 90° 0′ *W*).

8. Two places have the same latitude ϕ, and their difference of longitude is 2λ. Two ships sail from one place to the other, one on the great circle track and one on their parallel of latitude. Derive a formula for the difference of the distances sailed.

9. Find the area of the New York State Survey triangle described in Art. 56, considered as very small. (In the computations of the survey log $r = 6.804595$ was used, r in meters.)

10. By the use of formula (26), Art. 57, find the true area of the spherical triangle of Art. 56.

11. The continent of Asia has nearly the shape of an equilateral triangle with vertices shown on an old map as East Cape, Cape Romania, Baba Promontory. The map gives each side as very nearly 4800 and the radius as 3440 nautical miles, and states that the area is found very nearly by considering it as a plane triangle. Find the error.

58. *The Celestial Sphere.*—To an observer on the earth the sky and the heavens appear to be a vast hemispherical bowl inverted over the earth, and no matter what the position of the earth in its orbit or of the observer on the earth, this great bowl or vault appears always and everywhere to be equally vast and spherical. All the stars and the other celestial objects appear to lie in the surface of this sphere of indefinitely great radius and to the observer his position appears always to be at the center. The stars indeed appear as mere bright points and only the sun and moon have a discernible disc.

For purposes of reference and for convenience in locating celestial objects, this apparent sphere is considered as an actual sphere and is called the *celestial sphere*. Its radius is considered as

indefinitely great in comparison with that of the earth, the earth is considered as a point at the center, and the celestial objects are considered as points on the surface. Certain reference points are located on it and certain great circles, diameters, and radii are pictured as drawn on and through it in order to apply the methods of spherical trigonometry to the exact location of celestial objects and to trace their motions on its surface.

Fig. 27 represents the *celestial sphere* as seen from the outside, and its center O is the position of the earth. The various points, lines, and circles in Fig. 27 are defined as follows.

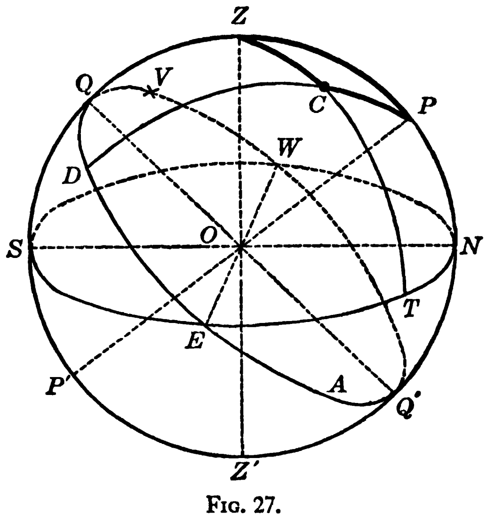


FIG. 27.

The point O being the position of the observer on the earth, Z is the point on the celestial sphere which is directly overhead and is called the *zenith*. It is the point where the line joining the earth's center and the observer's position meets the celestial sphere when extended. This line OZ is called the *vertical* at the observer's position. When the vertical is extended in the opposite direction it meets the celestial sphere at the point Z' which is called the ***nadir***.

The great circle $NESW$ is the intersection of the celestial sphere with the plane through O perpendicular to the vertical $Z'Z$. This plane is the observer's *horizontal* plane and the great circle $NESW$ is his *horizon*, his visible "edge of the world." The points N, E, S, W are the north, east, south, and west points of the horizon.

A *vertical circle* is a great circle on the sphere whose plane passes through the vertical OZ; the arc ZCT is a quadrant of a vertical circle.

The vertical circle $SZNZ'$ through the north and south points of the horizon is called the observer's *meridian*.

If C is a celestial object, such as a star, on the celestial sphere, the arc TC is its angular distance above the horizon measured on a vertical circle. $\overparen{TC}$ is called *altitude* or sometimes the *elevation* of the star or the point C.

The arc ZC is the complement of the altitude ($\overparen{ZC} = 90° - \overparen{TC}$) and is called the *zenith distance* of C.

The horizon arc NT, or the spherical angle NZT between the meridian NZS and the vertical circle ZCT is called the *azimuth* of the star or point C. The azimuth is also measured from the south point through the west to T as the arc $SWNT$.

The line PP' is the extended axis of the earth, meeting the celestial sphere in the points P, P', and these points are called the *poles* of the celestial sphere. In Fig. 27, P is the north pole. The north pole of the celestial sphere is not to be confused with the north point of the horizon. They are different points, but both are on the meridian $SZPN$.

As a consequence of the earth's rotation about the axis PP' the stars appear to describe small circles about P as their common center, apparently moving in the direction $EQWQ'$ or parallel to it.

The great circle $EQWQ'$ is the intersection of the celestial sphere with the plane through O perpendicular to PP', the earth's

equatorial plane, and the great circle $EQWQ'$ is called the *equator* of the celestial sphere, or the *celestial equator.*

The terrestrial *latitude* of the observer is the angular distance $\overset{\frown}{QZ}$ from the equator to the zenith. Since $\overset{\frown}{QP}=90°$ and $\overset{\frown}{NZ}=90°$, $\overset{\frown}{QZ}=\overset{\frown}{QP}-\overset{\frown}{ZP}=\overset{\frown}{NZ}-\overset{\frown}{ZP}=\overset{\frown}{NP}$; i.e., *the elevation of the pole above the horizon at any place (observer) is the latitude of the place.*

The north point (N) of the horizon will be on the meridian at different positions for observers at different latitudes, but the pole P is an invariable point because the axis PP' of the earth has always the same direction. There is a star, Polaris, very nearly at the point P and it is for this reason also called the Pole Star and the North Star. There is no visible star at or near the celestial south pole P'.

The great circle PCD through the pole and the star or point C is called the *hour circle* of C. All hour circles are perpendicular to the equator, and if the arc PCD is extended on the sphere it passes through P'.

The spherical angle ZPC at the pole, between the meridian and the hour circle, is called the *hour angle* of a star at C. It is measured from the meridian and is taken as positive toward the west (the direction of the apparent motion of the stars) and negative toward the east. Thus the hour angle ZPC in Fig. 27 is negative. The hour angle is equal to the angular measure of the arc QD on the equator.

The *hour angle* is so named because, if $ZPC=15°$, for example, one hour will elapse before PD will coincide with PQ in the apparent rotation of the celestial sphere about P, or before a star at C will appear on the meridian. Each star completes its apparent daily revolution about PP', or about the pole, in 24 hours and thus appears to pass over an arc of its path on the sphere equal to $360°\div 24=15°$ in each hour. Thus the total number of degrees in the hour angle divided by 15 gives the time before or after the star appears on the meridian.

The arc DC of the hour circle is the angular distance of C (star) from the equator and is called the *declination* of C. The declination is positive when C is north of the equator.

The angular distance VD along the equator toward the east, from a certain point V on the equator, is called the *right ascension* of the star; it is equal to the spherical angle at the pole between the hour circle of C and that of V. The angle between the hour circles of two stars is therefore equal to the difference between their right ascensions.

The point V is a fixed point on the equator; it is one of the two points where the equator meets the orbital plane of the earth (Art. 52). This plane meets the celestial sphere in a great circle which is called the *ecliptic*. The ecliptic is not shown in Fig. 27 but V and A represent the points where it intersects the equator. The point V is called the *vernal equinox* and A the *autumnal equinox*, but only V is used as a reference point on the equator and the ecliptic.

The line joining AV passes through O and is the line of intersection of the equatorial plane and the earth's orbital plane. Since the axis PP' is inclined to the orbital plane at an angle of $66° \, 32' \, 52''$ (Art. 52), and is perpendicular to the equatorial plane, the dihedral angle between the equatorial and orbital planes is $90° - 66° \, 32' \, 52'' = 23° \, 27' \, 8''$. The earth's orbital plane, whose intersection with the celestial sphere is the ecliptic, is also called the *ecliptic plane*, and the angle between it and the equatorial plane is called the *obliquity of the ecliptic* ($23° \, 27' \, 8''$).

From the definitions and explanations given above in connection with Fig. 27 and the celestial sphere, it is seen that the celestial pole and equator correspond to the terrestrial pole and equator, and the hour circle of the vernal equinox corresponds to the prime meridian of the terrestrial sphere (Art. 52). Therefore, the *declination DC* and the *right ascension VD* of a point C on the celestial sphere correspond to the latitude and longitude of a point on the terrestrial sphere, and are used in the same manner to locate points on the celestial sphere.

Similarly, the *altitude* TC and the *azimuth* NT or SWT, referred to the horizon and meridian, are also used to locate points on the celestial sphere.

The altitude and azimuth of a point, taken together, or the declination and right ascension taken together, are called the *spherical* or *celestial coordinates* of the point on the celestial sphere.

The ecliptic and a great circle through the vernal equinox V and the poles of the ecliptic (not P, P' and not shown in Fig. 27) are also sometimes used as reference circles, and angular distances measured on these to locate a point on the celestial sphere are also spherical coordinates. These angular distances are called the *celestial longitude and latitude*, respectively, but are not to be confused with the declination and right ascension referred to the equator and the vernal hour circle. In what follows, the celestial latitude and longitude will not be used.

The instruments and methods of observation and measurement used to determine the positions and celestial coordinates of celestial objects are treated in books on spherical, practical, and nautical astronomy and to a certain extent in books on geodesy and navigation. The methods of computation and of solution of problems are the methods of *spherical trigonometry.*

59. *The Astronomical Triangle.*—The spherical triangle ZPC in Fig. 27 is called the *astronomical triangle.* Its vertices are the zenith Z, the pole P, and the celestial object (star) C. The sides are ZP the co-latitude (since the elevation of the pole NP is equal to the latitude of the observer), the zenith distance ZC of the star, and its polar distance PC.

Fig. 27 is reproduced in part below as Fig. 28 and the astronomical triangle is lettered as in spherical trigonometry.

Referring to Fig. 28, the following notation will be used, C being a celestial object, say a star:

$$VD = \text{right ascension of the star} = \alpha$$

$$DC = \text{declination of the star} = \delta$$

$$QD = \angle ZPC = \text{hour angle of the star} = t$$

PC = polar distance of the star $= p$

ST = azimuth $= 180° \pm \overparen{NT}$ $= a$

TC = altitude of the star $= h$

ZC = zenith distance of the star $= z$

$QZ = NP$ = latitude of the observer $= \phi$

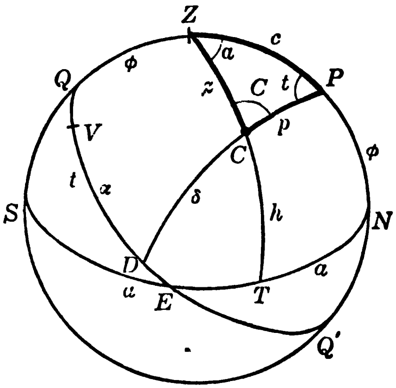


FIG. 28.

The sides and angles of the astronomical triangle are therefore

ZP = co-latitude of observer $= c$ $= 90° - \phi$,
PC = polar distance of star $= p$ $= 90° - \delta$,
ZC = zenith distance of star $= z$ $= 90° - h$;
ZPC = hour angle of star $= t$,
PZC = azimuth of star $= a$,
ZCP = angle at the star $= C$.

In using the astronomical triangle to solve practical problems the declination and right ascension (referred to equator and pole) are found in almanacs, and the altitude and azimuth (referred to the horizon and zenith) are found by observation. The astro-

nomical triangle is used in the solution of a few problems of practical and nautical astronomy in the last articles of this chapter.

From the definition of the hour angle given in Art. 58 in connection with Fig. 27, a very simple relation exists between the hour angle t of *the sun* and the local time at any place. When t is the absolute value and the sun is

$$\text{West of meridian, } time = \frac{t}{15} \text{ P.M.}$$

$$\text{East of meridian, } time = \left(12 - \frac{t}{15}\right) \text{ A.M.}$$

60. *Reduction of an Angle to the Horizon.*—This is an operation which is often necessary when the angular distance of two objects in space or on the celestial sphere is measured and the objects are not on the same vertical circle or any other convenient standard reference circle.

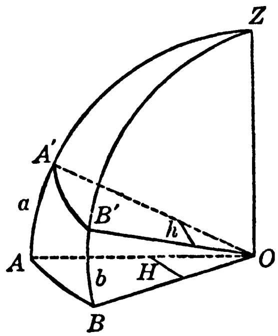


FIG. 29.

Thus in Fig. 29, A' and B' are the objects and O the point of observation. $\overset{\frown}{A'B'}$ is then the measured angular distance and $h = \angle A'OB'$ is the plane angle equal to $\overset{\frown}{A'B'}$. OZ is the vertical at O and Z the zenith; AB is an arc of the horizon between the vertical circles $AA'Z$, $BB'Z$ through A' and B'; and $H = \angle AOB$ = spherical $\angle AZB$ = arc $\overset{\frown}{AB}$ is the required angle. Let a, b be the elevations of A', B'.

In the spherical $\triangle A'ZB'$, $A'B' = h$, $A'Z = 90° - a$, $B'Z = 90° - b$ are known, and $\angle A'ZB' = \angle AZB = H$ is required. Thus the three sides are known and one angle is required. This is Case I, Art. 41, and the solution formulas used there are most convenient for the complete solution. When only one angle is desired, however, the

formula for $\cos\frac{H}{2}$ is more convenient than that for $\tan\frac{H}{2}$. This is formula (13), Art. 21. In the notation of Fig. 29 it is

$$\cos^2\frac{H}{2}=\frac{\sin\frac{1}{2}(90°-a+90°-b+h)\sin\frac{1}{2}(90°-a+90°-b-h)}{\sin(90°-a)\sin(90°-b)}$$

$$=\frac{\sin[90°-\frac{1}{2}(a+b-h)]\sin[90°-\frac{1}{2}(a+b+h)]}{\sin(90°-a)\sin(90°-b)}.$$

Let $\frac{1}{2}(a+b+h)=s$, then $\frac{1}{2}(a+b-h)=s-h$ and

$$\cos^2\frac{H}{2}=\frac{\sin[90°-(s-h)]\sin[90°-s]}{\sin(90°-a)\sin(90°-b)}=\frac{\cos(s-h)\cos s}{\cos a\cos b}.$$

$$\therefore\ \cos\frac{H}{2}=\sqrt{\frac{\cos(s-h)\cos s}{\cos a\cos b}}=\sqrt{\cos(s-h)\cos s\sec a\sec b}. \quad (28)$$

This formula gives the required horizontal projection H of the angular distance h in space between two objects whose elevations are a and b, with $s=\frac{1}{2}(a+b+h)$.

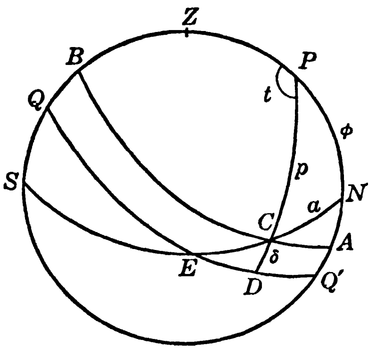


FIG. 30.

61. *Problem I.—To find the time of rising of a star (or the sun), given its declination and the latitude of the place of observation.*

Fig. 30 is a modification of Fig. 28. Here ACB is the apparent path of the star (or sun), a small circle parallel to the equator $Q'EQ$, and C is the point where it rises, i.e., crosses the horizon NES. $DC=\delta$ is the given declination and $NP=\phi$ is the latitude of the place of observation whose zenith is Z. The required time of rising (time to reach the meridian ZQS) is given by the hour angle ZPC.

In Fig. 30, therefore, spherical $\triangle CPN$ is right-angled at N; we have given

$$\begin{aligned} NP &= \text{latitude of the place} &&= \phi, \\ DC &= \text{declination of the star} &&= \delta, \\ PC &= \text{polar distance} &&= 90° - \delta; \end{aligned}$$

and it is required to find

$$\angle ZPD = \text{hour angle of star} \quad = t.$$

In right $\triangle CPN$, therefore, the *hypotenuse and leg* are given, and $\angle CPN$ is required. From Art. 32,

$$\cos \angle NPT = \frac{\tan \overset{\frown}{NP}}{\tan \overset{\frown}{PC}} = \tan \overset{\frown}{NP} \cot \overset{\frown}{PC}. \tag{29}$$

But $\angle NPC = 180° - \angle ZPC$, $\cos \angle NPC = -\cos \angle ZPC$, and hence (29) becomes

$$-\cos \angle ZPC = \tan \overset{\frown}{NP} \cot \overset{\frown}{PC};$$

and here $\cot \overset{\frown}{PC} = \cot p = \cot (90° - \delta) = \tan \delta$.

$$\therefore \quad \cos t = -\tan \phi \tan \delta. \tag{30}$$

This is the required solution formula, and when t is found by using the given values of ϕ, δ, then by Art. 59 the time is at once found. If the object C is the sun, it rises at $\left(12 - \frac{t}{15}\right)$ A.M.

62. *Problem II.—To find where a star (or the sun) will rise, given its declination and the latitude of the place of observation.*

Referring to Fig. 30, the data are the same as in Art. 61, and the same right $\triangle CPN$ is used. It is now required to find the point C where the star (or sun) crosses the horizon NS, and this is given by the azimuth $\overset{\frown}{NC} = a$.

In right $\triangle CPN$, therefore, the *hypotenuse and leg* are given, and side NC is required. By Art. 32, in the notation of Fig. 30,

$$\sin p = \cos a \cos \phi.$$

But $p = 90° - \delta$, $\sin p = \sin (90° - \delta) = \cos \delta$, and hence

$$\cos \delta = \cos a \cos \phi.$$

$$\therefore \quad \cos a = \frac{\cos \delta}{\cos \phi}. \tag{31}$$

By means of this solution formula the required a is found at once from the given δ, ϕ.

63. *Problem III.—To find the latitude of a place, given the time and the altitude and declination of a star as observed at that place.*

Referring to Fig. 28, for the place whose zenith is Z, we have given in this problem the hour $\angle ZPC = t =$ (the time) $\times 15°$, the altitude of the star $TC = h$, and its declination $DC = \delta$; and the latitude $QZ = NP = \phi$ is required.

In the *astronomical triangle* ZPC, therefore, are given $\angle ZPC = t$, side $ZC = z = 90° - h$, side $PC = p = 90° - \delta$; and side $ZP = c = 90° - \phi$ is required. This is Case V, Art. 46, given *two sides and one opposite angle*, required one side.

In the notation of Fig. 28, therefore,

$$\left.\begin{aligned} \sin a &= \frac{\sin t \sin p}{\sin z}, \\ \text{and} \qquad \tan \frac{c}{2} &= \frac{\sin \frac{1}{2}(a+t) \tan \frac{1}{2}(p-z)}{\sin \frac{1}{2}(a-t)}. \end{aligned}\right\} \tag{32}$$

But $\sin p = \sin (90° - \delta) = \cos \delta$, $\sin z = \sin (90° - h) = \cos h$, and $p - z = (90° - \delta) - (90° - h) = h - \delta$; and these values in (32) give

$$\sin a = \frac{\sin t \cos \delta}{\cos h}, \tag{33}$$

$$\tan\frac{c}{2}=\frac{\sin\frac{1}{2}(a+t)}{\sin\frac{1}{2}(a-t)}\tan\tfrac{1}{2}(h-\delta), \tag{34}$$

and finally $$\phi=90°-c. \tag{35}$$

These are the solution formulas of the problem. With the given hour angle t, declination δ, and altitude h, the azimuth a is found by (33); $a+t$ and $a-t$ are now known, and these in (34) with $h-\delta$ give $\frac{c}{2}$; ϕ is then the complement of c (35).

In using (34), $a-t$ or $t-a$ and $h-\delta$ or $\delta-h$ are to be used so that $\tan\frac{c}{2}$ will have the proper sign, to determine c so that ϕ will come out $+$ or $-$ in (35) according to the location of the place, north or south of the equator.

It is to be noted that the solution of this problem also gives the azimuth (a) of the star at the time of the observation.

64. *Problem IV.—To find the time, given the latitude of the place and the declination and altitude of a star (or the sun).*

Referring to Fig. 28, for the place whose zenith is Z, we have given the latitude $QZ=NP=\phi$ of the place, and the declination $DC=\delta$ and altitude $TC=h$ of the star (or sun); and the hour $\angle ZPC=t$ is required to find the time when the observation is made.

In the astronomical $\triangle ZPC$, therefore, are given $ZP=90°-\phi=c$, $PC=90°-\delta=p$, $CZ=90°-h=z$; and $\angle ZPC=t$ is required. This is Case I, Art. 41, *three sides* given, *one angle* required. Instead of the complete set of solution formulas which give all three angles, therefore, one of the sine or cosine formulas of (12) or (13) of Art. 21 is more suitable. We use here one of (12). In the notation of Fig. 28

$$\sin^2\frac{t}{2}=\frac{\sin(s-p)\sin(s-c)}{\sin p\sin c}. \tag{36}$$

In this $$2s=z+p+c$$

$$\therefore\ 2s=(90^\circ-h)+p+(90^\circ-\phi)$$
$$=180^\circ-(h-p+\phi).$$
$$\therefore\ s=90^\circ-\tfrac{1}{2}(h-p+\phi),$$
$$s-p=90^\circ-\tfrac{1}{2}(\phi+p+h),$$
$$s-c=\tfrac{1}{2}(\phi+p-h).$$

Therefore,
$$\sin(s-p)=\cos\tfrac{1}{2}(\phi+p+h)$$
$$\sin(s-c)=\sin\tfrac{1}{2}(\phi+p-h)$$
$$\sin p=\sin(90^\circ-\delta)=\cos\delta,$$
$$\sin c=\sin(90^\circ-\phi)=\cos\phi.$$

These values in (36) give

$$\sin^2\frac{t}{2}=\frac{\cos\frac{1}{2}(\phi+p+h)\sin\frac{1}{2}(\phi+p-h)}{\cos\phi\sin p}.$$

$$\therefore\ \sin\frac{t}{2}=\sqrt{\frac{\cos\frac{1}{2}(\phi+p+h)\sin\frac{1}{2}(\phi+p-h)}{\cos\phi\sin p}}. \qquad (37)$$

This is the required solution formula for the hour angle t, from which the time is found. In it, ϕ, h are the given latitude and star (sun) altitude, and the zenith distance $p=90^\circ-\delta$ is the complement of the given declination.

65. *Problem V.—To locate a star, given the latitude and the time, and the declination of the star.*

When the time is given the hour angle is known, and to locate a star for any observer is to determine its altitude and azimuth referred to his horizon and zenith. In Fig. 28, therefore, we have given the observer's latitude $QZ=NP=\phi$, the star declination $DC=\delta$, and hour $\angle ZPC=t$; and it is required to find its altitude $TC=h$ and azimuth $NT=a$.

In the astronomical $\triangle ZPC$, therefore, there are given side $ZP=90^\circ-\phi=c$, side $PC=90^\circ-\delta=p$, $\angle ZPC=t$; and the side $CZ=$

$90°-h=z$ and $\angle PZC=a$ are required. This is Case III, Art. 43, given *two sides and included angle*, but only one other side and angle are required, and not the complete solution. Instead of the complete solution formulas of Art. 43, therefore, the haversine formula (17) of Art. 44 and one of the sine proportions (4) of Art. 18 may be used.

In the notation of the astronomical triangle, Fig. 28, these are

$$\text{hav } z=\text{hav}(p-c)+\sin p \sin c \text{ hav } t, \tag{38}$$

and

$$\frac{\sin a}{\sin p}=\frac{\sin z}{\sin t},$$

or

$$\sin a=\frac{\sin p \sin z}{\sin t}. \tag{39}$$

But $p=90°-\delta, c=90°-\phi$; hence $p-c=(90°-\delta)-(90°-\phi)=\phi-\delta$, $\sin p=\cos \delta$, $\sin c=\cos \phi$; and these values in (38), (39) give

$$\text{hav } z=\text{hav }(\phi-\delta)+\cos \phi \cos \delta \text{ hav } t, \tag{40}$$

$$\sin a=\frac{\cos \delta \sin z}{\sin t}=\cos \delta \sin z \csc t. \tag{41}$$

These are the required solution formulas. By using the given latitude ϕ, declination δ, hour angle t (time) in (40), side z is found immediately; and this value of z in (41) with δ, t give the azimuth a at once. The required altitude is $h=90°-z$.

In using (40), it is to be noted that the quantity $Q=\cos \phi \cos \delta \text{ hav } t$, using cosines instead of the sines in (18), Art. 44.

Exercises

1. The sun's declination on July 4, 1881, was 22° 52′ 1″. At what time did the sun rise in latitude 40° 36′ 24″?

2. If the latitude of New York City is taken as 40° 42′ *N*, where will the star Arcturus rise on the horizon on a day when its declination is 19° 57′ *N*? (This is the star whose rays were used to turn on the lights of the Chicago World's Fair in 1933.)

3. On a certain day the declination of a star is 7° 54′, and at 3 : 2 : 28 P.M. local time (hour angle, +45° 42′) the altitude of the star is measured and found to be 22° 45′ 12″. Find the latitude of the place.

4. At a place in latitude 40° 36′ 24″ *N*, when the almanac showed the declination of a certain star to be 23° 4′ 24.3″, its altitude was measured as 47° 15′ 18″. Find the local time.

5. Where will a New York observer (40° 42′ *N*) find the star Regulus when it is 3 hours east of the meridian ($t = -45°$), on a day when its declination is 16° 3′ *N*?

6. Find the time of sunrise in latitude 40° 43′ 48″ *N* on the longest day of the year, the sun's declination (its greatest) being 23° 27′ *N*.

7. Find the length of the longest day of the year in north latitude 42° 16′ 48″.

8. Find the length of the shortest day of the year in north latitude 40° 29′ 52″, the sun being on that day its farthest south of the equator (declination 23° 27′ S).

9. Find the time of sunset in latitude 39° 6′ *N* on the winter day when the sun's declination is 15° 56′ *S*.

10. The great circle distance between Paris and Berlin is 472 nautical miles. The latitude of Paris is 48° 50′ 13″ *N*; Berlin, 52° 30′ 16″ *N*. When it is noon at Paris, what time is it at Berlin (exact local time).

11. Find the latitude of the place at which the sun rises exactly in the northeast on the longest day of the year.

12. At a place where the elevation of the north pole is 45°, the azimuth of a certain star on the horizon is 45° from the north. What is the polar distance of the star?

13. The oblique angle in space between the tops of two mountain peaks is 20° 45′; and their elevations are 15° 20′ and 22° 16′. What is the horizontal angle between the peaks at the point of observation.

14. The angular distance between two planets is 30° when their elevations are 32° and 40°. What is the difference of their azimuths?

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ANSWERS TO EXERCISES

Article 31, Page 65

1. $a = 118° 46' 6''$
   $b = 41° 42' 42''$
   $\alpha = 111° 7' 12''$.
2. $a = 52° 41' 42''$
   $b = 65° 19' 15''$
   $\beta = 69° 55' 20''$.
3. $a = 36° 27'$
   $b = 43° 32' 31''$
   $\beta = 57° 59' 19''$.
4. $a = 86° 40'$
   $b = 32° 40'$
   $\alpha = 88° 11' 58''$.
5. $a = 50°$
   $b = 56° 50' 49''$
   $\beta = 63° 25' 4''$.
6. $a = 50°$
   $b = 127° 4' 30''$
   $\beta = 120° 3' 50''$.
7. $a = 26° 27' 24''$
   $b = 39° 57' 42''$
   $\beta = 62° 0' 4''$.
8. $a = 136° 15' 32''$
   $b = 48° 23' 38''$
   $\beta = 58° 27' 4''$.
9. $a = 67° 6' 53''$
   $b = 155° 46' 43''$
   $\beta = 153° 28' 24''$.
10. $a = 54° 41' 35''$
   $b = 104° 21' 28''$
   $\alpha = 55° 32' 45''$.
11. $a = 34° 6' 13''$
   $b = 87° 32' 40''$
   $\beta = 88° 37' 42''$.
12. $a = 37° 40' 12''$
   $b = 0° 26' 36''$
   $\alpha = 89° 25' 32''$.

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1. $c = 54° 20'$
   $\alpha = 46° 59' 43''$
   $\beta = 57° 59' 19''$.
2. $c = 87° 11' 40''$
   $\alpha = 88° 11' 58''$
   $\beta = 32° 42' 39''$.
3. $c = 59° 4' 26''$
   $\alpha = 63° 15' 13''$
   $\beta = 44° 26' 22''$.
4. $c = 63° 55' 43''$
   $\alpha = 105° 44' 21''$
   $\beta = 147° 19' 47''$.
5. $b = 51° 53'$
   $\alpha = 27° 28' 38''$
   $\beta = 73° 27' 11''$.
6. $b = 19° 17'$
   $\alpha = 37° 36' 49''$
   $\beta = 54° 49' 23''$.
7. $b = 32° 41'$
   $\alpha = 49° 20' 16''$
   $\beta = 50° 19' 16''$.
8. $b = 79° 13' 38''$
   $\alpha = 131° 43' 50''$
   $\beta = 81° 58' 53''$.
9. $b = 28° 14' 31''$
   $c = 78° 53' 20''$
   $\beta = 28° 49' 57''$.
   $b' = 151° 45' 29''$
   $c' = 101° 6' 40''$
   $\beta' = 151° 10' 3''$.
10. Impossible.
11. $b = 49° 59' 58''$
   $c = 91° 47' 40''$
   $\alpha = 92° 8' 23''$.
12. $b = 0° 27' 10''$
   $c = 2° 3' 56''$
   $\alpha = 77° 20' 28''$.
13. $b = 15° 16' 50''$
   $c = 25° 14' 38''$
   $\alpha = 54° 35' 17''$.
14. $b = 30° 8' 39''$
   $c = 59° 51' 21''$
   $\alpha = 70° 17' 35''$.
15. $a = 50° 0' 4''$
   $b = 143° 5' 12''$
   $c = 120° 55' 34''$.
16. $a = 120° 10' 3''$
   $b = 119° 59' 46''$
   $c = 75° 26' 58''$.
17. $a = 36° 27'$
   $b = 43° 32' 30''$
   $c = 54° 20'$.
18. $a = 90°$
   $b = 88° 24' 35''$
   $c = 90°$.

19. $b=72^\circ\ 2'\ 48''$
$\alpha=34^\circ\ 15'$
$\beta=80^\circ\ 0'\ 36''$.

20. $a=103^\circ\ 28'\ 6''$
$\alpha=98^\circ\ 50'\ 30''$
$\beta=41^\circ\ 17'\ 42''$.

21. $a=25^\circ\ 12'\ 48''$
$b=52^\circ\ 0'\ 45''$
$c=56^\circ\ 9'\ 36''$.

22. $a=131^\circ\ 7'$
$b=103^\circ\ 24'\ 30''$
$c=81^\circ\ 13'\ 42''$.

23. $c=78^\circ\ 35'\ 18''$
$\alpha=44^\circ\ 26'$
$\beta=79^\circ\ 1'\ 24''$.

24. $c=88^\circ\ 33'\ 30''$
$\alpha=99^\circ\ 53'\ 48''$
$\beta=98^\circ\ 12'\ 30''$.

25. $a=47^\circ\ 49'\ 30''$
$c=74^\circ\ 27'\ 36''$
$\beta=72^\circ\ 7'\ 30''$.

26. $b=161^\circ\ 3'\ 12''$
$c=149^\circ\ 15'$
$\alpha=54^\circ\ 45'\ 36''$.

27. $b=53^\circ\ 41'\ 54''$
$\alpha=46^\circ\ 52'\ 15''$
$\beta=64^\circ\ 24'$.

28. $b=23^\circ\ 57'$
$c=72^\circ\ 1'\ 15''$
$\beta=25^\circ\ 15'\ 42''$.
$b'=156^\circ\ 3'$
$c'=107^\circ\ 58'\ 45''$
$\beta'=154^\circ\ 44'\ 18''$.

29. $a=20^\circ\ 46'$
$c=36^\circ\ 21'\ 36''$
$\beta=58^\circ\ 59'\ 42''$.

30. $a=29^\circ\ 5'\ 36''$
$b=69^\circ\ 29'$
$\beta=79^\circ\ 41'\ 15''$.

31. $a=121^\circ\ 23'\ 36''$
$b=29^\circ\ 11'$
$c=117^\circ\ 3'$.

32. $c=107^\circ\ 50'\ 12''$
$\alpha=30^\circ\ 23'\ 6''$
$\beta=100^\circ\ 10'\ 54''$.

33. $a=126^\circ\ 45'\ 54''$
$b=167^\circ\ 45'\ 12''$
$\alpha=99^\circ\ 0'\ 18''$.

34. $b=66^\circ\ 43'\ 30''$
$c=72^\circ\ 25'$
$\alpha=42^\circ\ 32'\ 42''$.

35. $a=127^\circ\ 8'\ 6''$
$b=68^\circ\ 49'$
$\beta=72^\circ\ 49'\ 48''$.

36. $a=35^\circ\ 4'\ 24''$
$b=54^\circ\ 42'$
$c=61^\circ\ 46'\ 36''$.

37. $a=66^\circ\ 14'\ 6''$
$\alpha=82^\circ\ 45'\ 45''$
$\beta=161^\circ\ 46'\ 54''$.

38. $c=67^\circ\ 2'\ 48''$
$\alpha=110^\circ\ 39'\ 42''$
$\beta=135^\circ\ 57'\ 42''$.

39. $a=166^\circ\ 29'\ 30''$
$b=131^\circ\ 1'\ 42''$
$\alpha=162^\circ\ 20'\ 6''$.

40. $a=39^\circ\ 43'\ 54''$
$c=40^\circ\ 48'\ 6''$
$\alpha=78^\circ\ 0'\ 42''$.
$a'=140^\circ\ 16'\ 6''$
$c'=139^\circ\ 11'\ 54''$
$\alpha'=101^\circ\ 59'\ 18''$.

41. $a=150^\circ\ 26'\ 12''$
$b=94^\circ\ 43'\ 30''$
$\beta=99^\circ\ 36'\ 36''$.

42. $b=108^\circ\ 51'\ 6''$
$\alpha=123^\circ\ 30'\ 45''$
$\gamma=102^\circ\ 4'\ 42''$.

43. $\alpha=142^\circ\ 15'\ 12''$
$\beta=117^\circ\ 50'\ 15''$
$\gamma=111^\circ\ 40'\ 6''$.

44. $b=56^\circ\ 39'\ 36''$
$\alpha=116^\circ\ 46'\ 24''$
$\beta=53^\circ\ 36'\ 54''$.

45. $b=14^\circ\ 6'\ 12''$
$\beta=13^\circ\ 25'\ 18''$
$\gamma=72^\circ\ 17'\ 30''$.
$b'=165^\circ\ 43'\ 48''$
$\beta'=166^\circ\ 34'\ 42''$
$\gamma'=107^\circ\ 42'\ 30''$.

Article 41, Page 76

No	α	β	γ	No	α	β	γ
1.	116° 44′ 50″	63° 15′ 10″	91° 7′ 22″	**10.**	138° 15′ 45″	31° 11′ 14″	35° 49′ 58″
2.	59° 4′ 28″	94° 23′ 12″	120° 4′ 52″	**11.**	128° 44′ 45″	33° 11′ 12″	18° 15′ 31″
3.	132° 14′ 21″	110° 10′ 40″	99° 42′ 24″	**12.**	110° 51′ 16″	48° 56′ 4″	38° 26′ 48″
4.	20° 9′ 55″	55° 52′ 35″	114° 20′ 21″	**13.**	142° 11′ 38″	120° 15′ 57″	113° 28′ 2″
5.	127° 22′ 7″	51° 18′ 12″	72° 26′ 40″	**14.**	121° 36′ 20″	42° 15′ 13″	34° 15′ 3″
6.	120° 3′ 36″	46° 17′ 12″	31° 38′ 6″	**15.**	47° 51′ 50″	123° 53′ 48″	57° 46′ 56″
7.	120° 2′ 36″	74° 1′ 36″	98° 13′ 36″	**16.**	143° 18′ 34″	111° 3′ 18″	131° 29′ 32″
8.	95° 50′ 0″	48° 17′ 0″	39° 37′ 30″	**17.**	26° 34′ 55″	52° 7′ 48″	125° 7′ 57″
9.	145° 16′ 0″	136° 19′ 12″	132° 41′ 0″	**18.**	83° 12′ 10″	114° 30′ 0″	123° 20′ 32″

Article 42, Page 78

No	a	b	c	No.	a	b	c
1.	135° 49′ 20″	144° 37′ 15″	60° 4′ 54″	**7.**	31° 9′ 13″	84° 18′ 28″	115° 10′ 0″
2.	67° 25′ 35″	143° 44′ 46″	132° 10′ 26″	**8.**	139° 21′ 22″	126° 57′ 52″	56° 51′ 48″
3.	89° 16′ 53″	52° 39′ 5″	112° 22′ 59″	**9.**	83° 52′ 36″	115° 55′ 0″	137° 18′ 48″
4.	125° 19′ 50″	62° 54′ 16″	131° 23′ 32″	**10.**	119° 50′ 24″	81° 20′ 12″	86° 46′ 48″
5.	51° 17′ 31″	64° 2′ 47″	51° 17′ 31″	**11.**	64° 47′ 12″	61° 47′ 24″	48° 3′ 24″
6.	104° 25′ 9″	53° 49′ 25″	97° 44′ 19″	**12.**	98° 44′ 48″	83° 25′ 0″	75° 23′ 12″

Article 45, Page 83

1. $\beta=135° 5' 29''$
   $\gamma=50° 30' 8''$
   $a=69° 34' 56''$.
2. $\beta=95° 38' 4''$
   $\gamma=97° 26' 29''$
   $a=64° 23' 15''$.
3. $\alpha=56° 16' 15''$
   $\beta=45° 4' 41''$
   $c=96° 20' 44''$.
4. $\alpha=130° 5' 22''$
   $\beta=32° 26' 6''$
   $c=51° 6' 12''$.
5. $\alpha=70° 39' 3''$
   $\gamma=48° 35' 59''$
   $b=112° 23' 2''$.
6. $\alpha=90° 43' 7''$
   $\beta=67° 37' 1''$
   $c=131° 24' 0''$.
7. $\alpha=116° 9' 6''$
   $\beta=35° 46' 15''$
   $c=51° 2' 24''$.
8. $\alpha=63° 15' 11''$
   $\beta=132° 17' 58''$
   $c=59° 4' 17''$.
9. $\alpha=129° 58' 2''$
   $\beta=63° 15' 8''$
   $c=55° 52' 40''$.
10. $\beta=88° 12' 24''$
    $\gamma=55° 52' 42''$
    $a=50° 1' 40''$.
11. $\beta=56° 11' 57''$
    $\gamma=123° 21' 12''$
    $a=67° 11' 47''$.
12. $\alpha=98° 56'$
    $\beta=66° 18'$
    $c=103° 30' 36''$.
13. $a=120° 30' 30''$
    $c=70° 20' 20''$
    $\beta=50° 10' 10''$.
14. $a=99° 40' 48''$
    $c=100° 49' 30''$
    $\beta=65° 33' 10''$.
15. $a=84° 14' 29''$
    $b=44° 13' 45''$
    $\gamma=36° 45' 28''$.
16. $a=104° 34' 28''$
    $b=55° 25' 32''$
    $\gamma=124° 42' 0''$.
17. $a=63° 39' 58''$
    $b=75° 0' 52''$
    $\gamma=42° 30' 55''$.
18. $a=70° 20' 50''$
    $b=38° 27' 59''$
    $\gamma=52° 30' 20''$.
19. $a=37° 14' 10''$
    $b=121° 28' 10''$
    $\gamma=161° 22' 11''$.
20. $a=125° 41' 43''$
    $b=82° 47' 34''$
    $\gamma=127° 22' 0''$.
21. $b=152° 43' 51''$
    $c=88° 12' 21''$
    $\alpha=78° 15' 48''$.
22. $a=128° 41' 46''$
    $c=107° 33' 20''$
    $\beta=55° 47' 40''$.
23. $b=145° 55' 12''$
    $c=119° 22' 30''$
    $\alpha=80° 15' 0''$.
24. $a=37° 58' 54''$
    $b=11° 52' 54''$
    $\gamma=29° 13' 24''$.

Article 47, Page 87

1. $\beta = 36°\ 29'\ 46''$
   $\gamma = 51°\ 37'\ 56''$
   $c = 43°\ 7'\ 14''$.

2. $\beta = 116°\ 42'\ 30''$
   $\gamma = 116°\ 47'$
   $c = 120°\ 57'\ 27''$.

3. $\beta = 120°\ 47'\ 45''$
   $\gamma = 97°\ 42'\ 55''$
   $c = 55°\ 42'\ 8''$
   $\beta' = 59°\ 12'\ 15''$
   $\gamma' = 29°\ 8'\ 39''$
   $c' = 23°\ 57'\ 17''$

4. $\beta = 90°$
   $\gamma = 45°\ 44'\ 5''$
   $c = 45°\ 12'\ 19''$.

5. Impossible.

6. $a = 47°\ 8'\ 40''$
   $\alpha = 98°\ 53'\ 24''$
   $\gamma = 27°\ 52'\ 54''$.

7. $a = 151°\ 45'\ 12''$
   $\alpha = 152°\ 2'\ 12''$
   $\gamma = 72°\ 49'\ 15''$.

8. $\alpha = 65°\ 33'\ 10''$
   $\beta = 97°\ 26'\ 29''$
   $b = 100°\ 49'\ 30''$.

9. $a = 153°\ 38'\ 42''$
   $\alpha = 160°\ 1'\ 24''$
   $\gamma = 42°\ 37'\ 18''$.
   $a' = 90°\ 5'\ 41''$
   $\alpha' = 50°\ 18'\ 55''$
   $\gamma' = 137°\ 22'\ 42''$.

10. Impossible.

11. $c = 100°\ 49'\ 28''$
    $\beta = 65°\ 33'\ 9''$
    $\gamma = 97°\ 26'\ 26''$.

12. $c = 95°\ 18'\ 17''$
    $\beta = 57°\ 34'\ 51''$
    $\gamma = 115°\ 57'\ 51''$.
    $c' = 28°\ 45'\ 5''$
    $\beta' = 122°\ 25'\ 9''$
    $\gamma' = 25°\ 44'\ 32''$.

13. $b = 155°\ 5'\ 18''$
    $c = 33°\ 1'\ 37''$
    $\gamma = 70°\ 20'\ 40''$.

14. $b = 124°\ 7'\ 20''$
    $c = 159°\ 50'\ 15''$
    $\gamma = 159°\ 43'\ 34''$.

15. $b = 90°$
    $c = 147°\ 41'\ 50''$
    $\gamma = 148°\ 5'\ 40''$.

16. Impossible.

17. $b = 51°\ 17'\ 54''$
    $c = 41°\ 4'\ 42''$
    $\gamma = 36°\ 38'\ 48''$.

18. $b = 54°\ 36'\ 48''$
    $c = 147°\ 36'\ 54''$
    $\gamma = 139°\ 39'\ 6''$.
    $b' = 125°\ 23'\ 12''$
    $c' = 6°\ 51'\ 6''$
    $\gamma' = 9°\ 17'\ 36''$.

19. $a = 114°\ 26'\ 50''$
    $c = 82°\ 33'\ 31''$
    $\gamma = 79°\ 10'\ 30''$.

20. $b = 36°\ 5'\ 34''$
    $c = 50°\ 24'\ 57''$
    $\gamma = 70°\ 55'\ 35''$.

21. $a = 42°\ 37'\ 18''$
    $c = 129°\ 41'\ 5''$
    $\gamma = 89°\ 54'\ 19''$.
    $a' = 137°\ 22'\ 42''$
    $c' = 19°\ 58'\ 36''$
    $\gamma' = 26°\ 21'\ 18''$.

22. Impossible.

23. $b = 63°\ 39'\ 58''$
    $c = 41°\ 9'\ 46''$
    $\gamma = 42°\ 30'\ 55''$.

24. $b = 80°\ 19'\ 9''$
    $c = 120°\ 48'\ 5''$
    $\gamma = 131°\ 29'\ 53''$
    $b' = 99°\ 40'\ 51''$
    $c' = 151°\ 27'\ 3''$
    $\gamma' = 155°\ 22'\ 19''$.

Article 51, Page 96

4. $3e^2$.
5. $e^3\left(\frac{\sqrt{3}-1}{2}\right)^{\frac{3}{2}}$.
6. $6+4(\sqrt{2}+\sqrt{6})$.
7. $\sqrt[4]{972}$.
8. $E=70°\ 31'\ 43''$.
9. $90°$.
10. $109°\ 28'\ 16''$.
11. $116°\ 33'\ 54''$.
12. $138°\ 11'\ 23''$.
13. $\cos E=\frac{1}{3}$.
14. 0.
15. $-\frac{1}{3}$.
16. $-\frac{1}{5}\sqrt{5}$.
17. $-\frac{1}{3}\sqrt{5}$.
18. $S=1.7321a^2$.
19. $6a^2$.
20. $3.46a^2$.
21. $20.646a^2$.
22. $8.6603a^2$.
23. $V=0.1179a^3$.
24. a^3.
25. $0.4714a^3$.
26. $7.6637a^3$.
27. $2.1817a^3$.

Article 57, Page 115

1. $69°\ 48'=4774$ land mi.
2. $52°\ 27'=3624$ land mi.
4. (*a*) $56°\ 59'$. (*b*) $51°\ 27'\ N$, $5°\ 50'\ W$. (*c*) $28°\ 42'$; 1722 mi.
5. $D=74°\ 31'=4470$ mi.
6. Course from Vancouver, $S\ 50°\ 17'\ W$; Honolulu, $N\ 32°\ 3'\ E$; vertex ($60°\ 22'\ N$, $80°\ 41'\ W$); distance, 2698 miles.
7. $\overset{\frown}{AB}=116°\ 58'=7018$ naut. mi.
8. Difference$=2r[\lambda\cos\phi-\sin^{-1}(\cos\phi\cos\lambda)]$.
9. $K'=527783000$ sq. meters.
10. $K=527784000$ sq. meters.
11. Plane area 9,976,500; spherical area 13,316,560 sq. naut. miles.

Article 65, Page 129

1. $t=111°\ 11'\ 44''=$ 7:24:47 A.M.
2. $a=63°\ 15'\ 11''$ from North.
3. $\phi=67°\ 58'\ 54''\ N$.
4. $t=+46°\ 40'\ 5''=$ 3:6:40 P.M.
5. Azimuth$=27°\ 18'\ 40''\ E$ from S; altitude$=44°\ 10'\ 33''$.
6. 4:32:16 A.M.
7. 15 hr. 5 min. 50 sec.
9. 5:6:0 P.M.
10. 12:44 P.M.
11. $\phi=55°\ 45'\ 6''\ N$.
12. $60°$.